Two Strategies for Approximate Computational Geometry

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joint work with

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Problem

- Algorithms are expressed in real RAM model.
- Input is assumed in general position.
- Implementations must use computer arithmetic.
- Implementations must handle degenerate input.

Geometric Predicates

- Main interface with real RAM model (also geometric constructions).
- Predicate $P(x)$ is true when polynomial $f(x)$ is positive.
- Unsafe predicate: $|f(x)|$ near the rounding unit.
- Degenerate predicate: $f(x) = 0$.
- Singular predicate: $f(x) = 0$ and $f'(x) = 0$.

Exact Computational Geometry

- Implement predicates exactly using real algebraic geometry.
- Symbolic perturbation of degenerate predicates.
- Technical Problems
  - Running time grows rapidly with algebraic degree.
  - Bit complexity grows rapidly in iterated computation.
  - Large constant factors and programming overhead.

Conceptual Problem

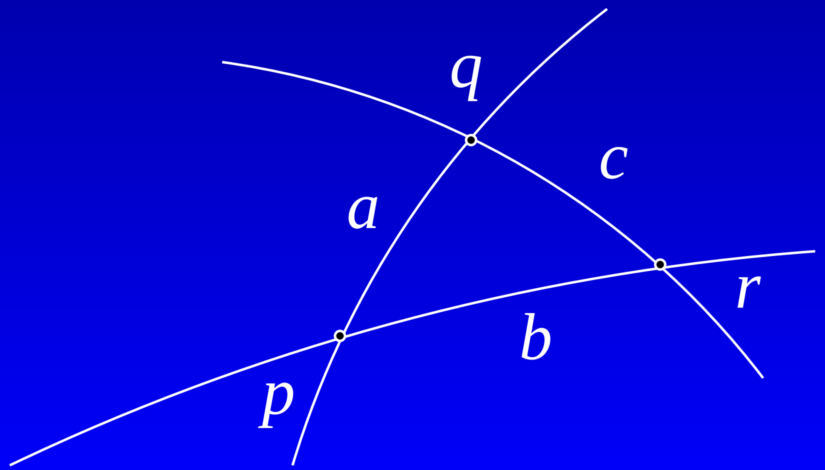
- Scientific computing is approximate because exact solutions are impractical and unnecessary.
- That is why rounding and numerical analysis were invented.
- Why should computational geometry be exact?

Approximate Comp. Geometry

- Implement predicates approximately using floating point arithmetic and numerical solvers.
- Advantages:
  - Running time grows modestly with degree.
  - Constant bit complexity.
  - Small constant factors.
- Challenge: generate consistent output.

Consistency

- Error metric: distance from input to perturbed input for which the computed output is correct.
- Inconsistent output: no such perturbation exists.
- Example: plane curves in cyclic vertical order.
- $a < b$ before p_x, $b < c$ before r_x, $c < a$ after q_x.
- Numerical error causes $q_x < p_x$.
- Inconsistency: $a < b < c < a$ on (q_x, p_x).



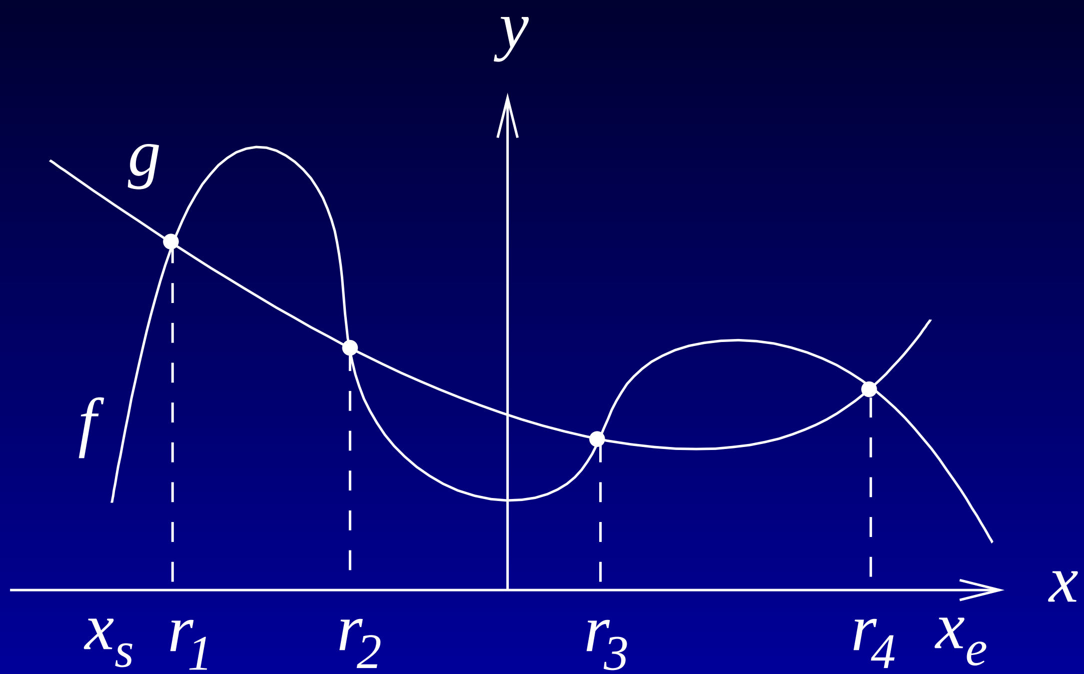
Inconsistency Sensitive Strategy

- Adapt RAM algorithms to generate consistent output despite computation error.
- Bound output error and extra cost in terms of computation error and inconsistency count.
- Advantage: speed and accuracy.
- Disadvantage: lack of generality.

Arrangement Algorithm

- Input: x-monotonic semi-algebraic curves, crossing module.
- Step 1: Compute curve crossings and y-order.
- Step 2: Embed curve endpoints.
- Output: $\mathrm{O}(\epsilon + kn\epsilon)$ accurate arrangement for n curves and an ϵ-accurate crossing module with k inconsistencies.
- Proving consistency is much easier than proving an error bound!

Crossing Module



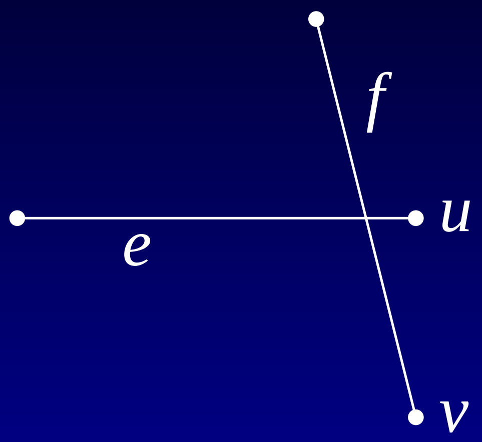
- Crossings computed with custom eigensolver.
- Accuracy, ϵ, of 12–16 decimal digits.
- Running time cd^4 for degree d.
- $c = 6$ microseconds on 2.2 GHz processor.

Step 1: Curve y-order

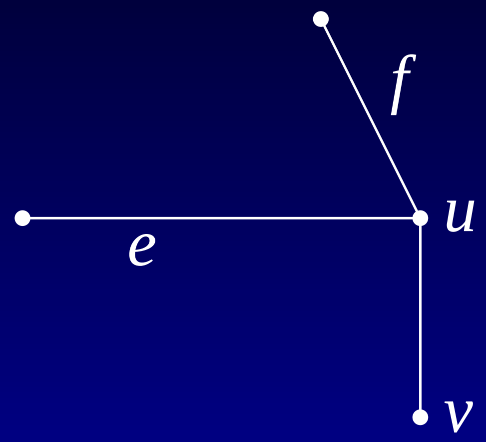
- Crossing module defines curve y-order, $a <_x b$.
- k inconsistencies: $a <_x b <_x c <_x a$ on maximal open interval.
- Bentley sweep with two modifications:
  1. Don't swap non-adjacent curves.
  2. Immediately swap out-of-order curves.
- Sweep list defines output y-order, $a <'_x b$.
- Error analysis: bound distance between a, b at x where $a <'_x b$ and $b <_x a$.
- Key idea: there exists a sequence $a <_x s_1 <_x \cdots <_x s_p <_x b$ with $p \leq k$.

Step 2: Endpoint Embedding

Inconsistency between endpoint and curve y-orders.



inconsistency



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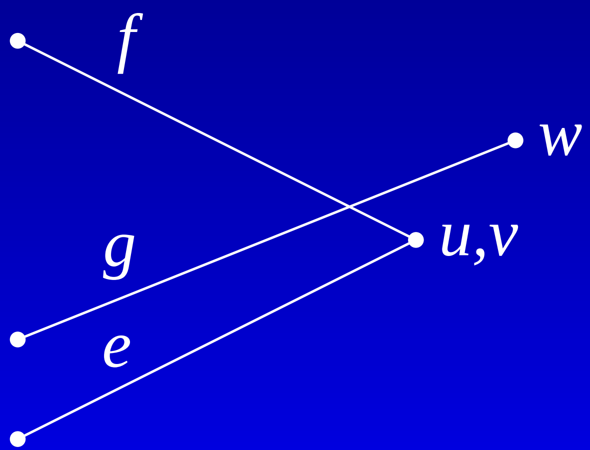

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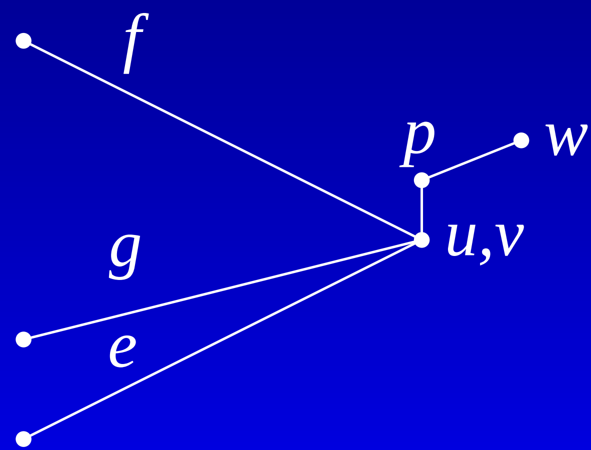

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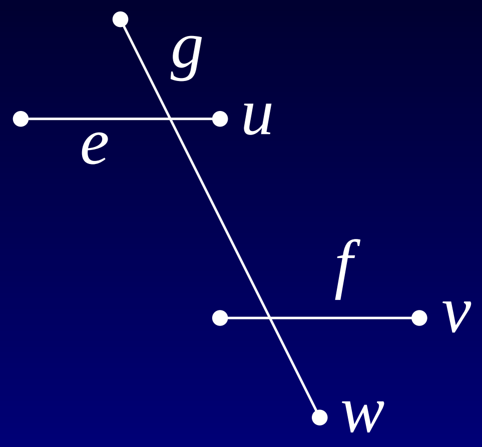

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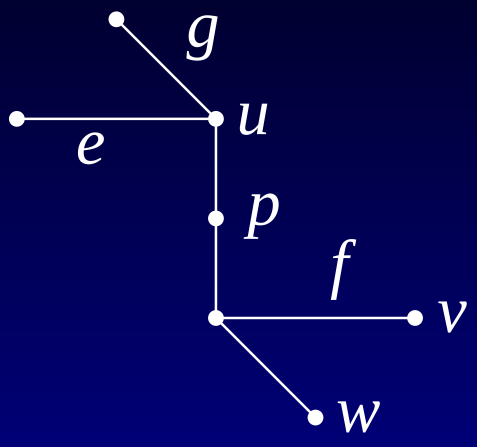

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Perturbation Methods

- Perturb input to avoid inconsistency and degeneracy.
- Minimize perturbation size relative to success probability.
- Advantage: general.
- Disadvantages of prior work
  - inaccurate, especially for near-singular input.
  - incompatible with equality constraints (implicit parameters).

Constrained Linear Perturb.

Strategy: assign signs to polynomials then compute minimal perturbation that realizes these signs.

- No error or cost for safe polynomials.
- Accurate perturbation of singular polynomials.
- Implicit parameters handled.
- Signs can be set to zero.

CLP Implementation Strategy

- Online algorithm: compute perturbation for both signs of polynomial subject to prior signs; select smaller perturbation.
- Linear programming implementation.
- Linear Taylor series for regular polynomials.
- Replace a near-singular polynomial with a regular proxy and constrain it to have the same sign.

CLP Definition

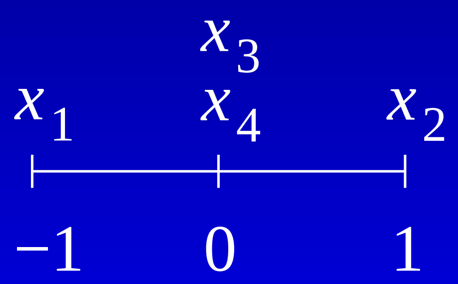
- CLP defined for polynomials $f_1, \ldots, f_m$ at $\mathbf{x} = \mathbf{a}$.
- f_i safe: $|f_i(\mathbf{a})| > k_i\mu$ with μ the rounding unit.
- Perturbation: $\mathbf{p} = \mathbf{a} + \delta\mathbf{v}$, $\delta \geq 0$, $||\mathbf{v}|| = 1$.
- CLP: $\mathbf{p}$ and signs $s_1, \ldots, s_m$ with $s_i = \pm 1$.
- If f_i is safe, s_i is the computed sign. If not, s_i is the sign of the rate of change $\nabla f_i \cdot \mathbf{v}$.
- $s_i f_i(\mathbf{p}) \geq k_i\mu$ for $i = 1, \ldots, m$.

Core Algorithm

- Extend CLP from $f_1, \ldots, f_{m-1}$ to f_m.
- If f_m is safe, return the computed sign and the prior $\mathbf{p}$.
- Else assign the sign and $\mathbf{v}$ that maximize the minimum of the rates, $r_i = s_i \nabla f_i \cdot \mathbf{v}/k_i$, at which the unsafe f_i become safe.
- Maximize r subject to $r_i \geq r$ and $s_m = \pm 1$; assign s_m and $\mathbf{v}$ from the larger r value.
- Set $\delta = 2\mu/r$ to make s_i correct for the linearized f_i with margin $2k_i\mu$.
- Verify $s_i f_i(\mathbf{p}) \geq k_i \mu$ for all unsafe f_i.

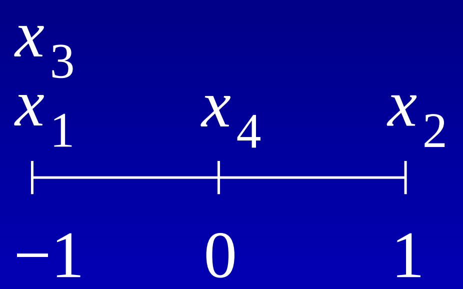
Sorting Example

- Sort four equal numbers $x_i = 0$.
- Predicate polynomials are $x_i - x_j$ with $k_i = 1$.
- Six signs assigned during sorting.
- Perturbation direction constraints: $-1 \le v_i \le 1$.
- Sign 1: $x_2 - x_1$ with cases $v_2 - v_1 \ge r$ and $v_1 - v_2 \ge r$; maximum of $r = 2$ for both, so $s_1 = 1$ and $x_1 < x_2$.

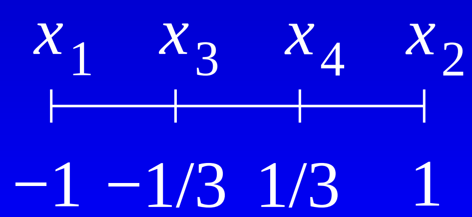


Sorting Example

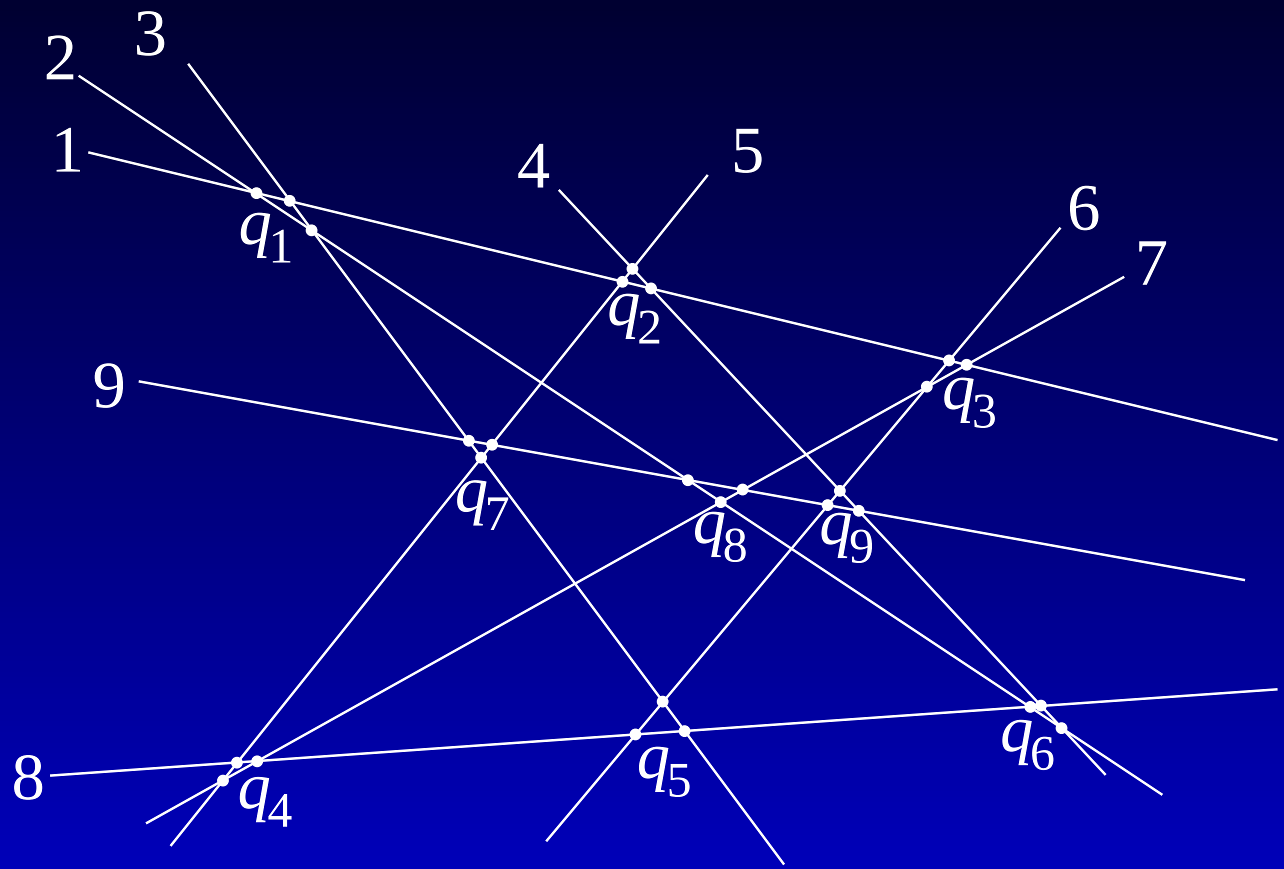
- Sign 2: $x_3 - x_2$
  - $s_2 = 1$: $v_3 - v_2 \geq r$ and $v_2 - v_1 \geq r$ with maximum $r = 1$ at $\mathbf{v} = (-1, 0, 1, 0)$.
  - $s_2 = -1$: $v_2 - v_3 \geq r$ and $v_2 - v_1 \geq r$ with maximum $r = 2$ at $\mathbf{v} = (-1, 1, -1, 0)$.
  - Set $s_2 = -1$ and $x_3 < x_2$.



- Sign 6: $x_1 < x_3 < x_4 < x_2$.



Pappus Example



- Sort x coordinates of the intersection points of 9 lines with 9 near-triple intersection points.
- First 8 triples permit all signs; 254 of these permit both signs for ninth triple.

Full CLP algorithm

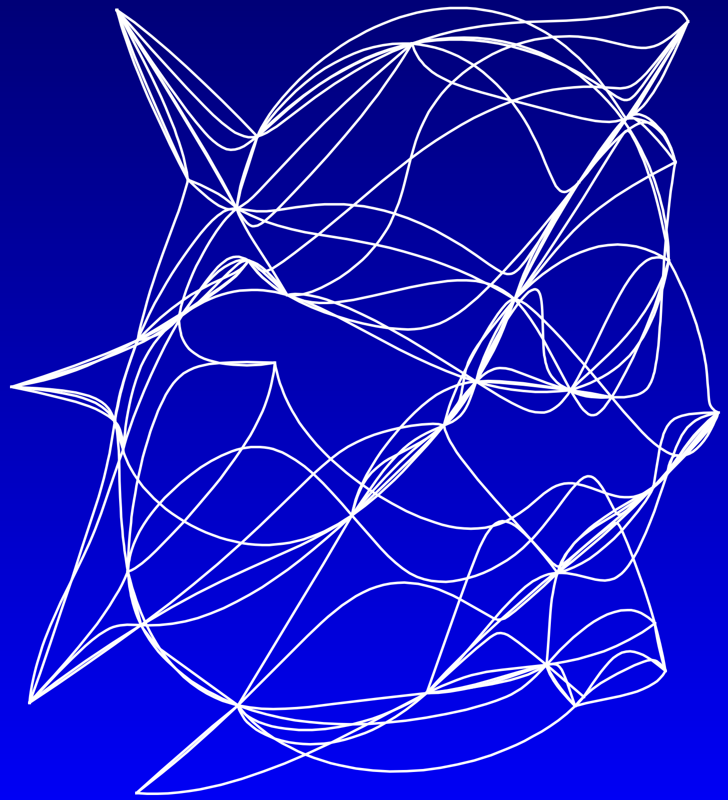
- Proxies for near-singular polynomials.
  - Status: manual construction.
  - Research: automated construction for determinant polynomials.
- Implicit parameter definitions.
  - Status: regular definitions.
  - Research: singular definitions.
- Output simplification.
- Random perturbation direction.

CLP versus controlled pert.

- Convex hull of n identical points: $\delta = 121\mu$ for $n = 100$, $\delta = 238\mu$ for $n = 200$, $\delta = 1619\mu$ for $n = 1000$.
- Controlled perturbations 2×10^8 times larger.
- Delaunay triangulation of n identical points: $\delta = 399\mu$ for $n = 100$, $\delta = 1767\mu$ for $n = 200$, $\delta = 14959\mu$ for $n = 1000$.
- Controlled perturbations 10^{11} times larger.
- Delaunay triangulation of n points on unit line segment: $\delta = 636\mu$ for $n = 100$, $\delta = 2933\mu$ for $n = 200$, $\delta = 8479\mu$ for $n = 1000$.
- Controlled perturbations 2×10^6, 7×10^6, 2×10^9 times larger.

CLP versus ECG

- Arrangement of 100 random degree-6 algebraic curves: 22 seconds with CLP; 220 seconds with ECG [Eigenwillig, 2008].
- Arrangement of 100 degenerate degree-6 semi-algebraic curves.



CLP versus ECG

- Arrangement contains 1330 vertices, including 43 clusters of nearly equal vertices with an average of 23 vertices per cluster and 55 vertices in the largest cluster.
- 1.5 seconds with CLP; estimated 30,000 seconds with ECG.
- Estimate based on measured root separation, ρ, and on published $\log^2 \rho$ running time.

Conclusion

- Approximate computational geometry is fast and accurate.
- Consistency is the challenge.
- Consistency sensitivity works case by case.
- CLP is algorithm-independent.
- We aim for a black box CLP library.