

Sign the exam with your student number - not your name _____

Answer the five questions posed below to the best of your ability.

1. (20 pts) Let $p(x, y)$ denote the statement: $y^2 - x = y + x^2$, where the universe for the variables x and y is the set of all integers. Determine the truth value of the following statements.

(a) $p(0, 0)$

(b) $(\exists y)(p(1, y))$

(c) $(\forall x)(\exists y)(p(x, y))$

(d) $(\forall x)(\forall y)(p(x, y))$

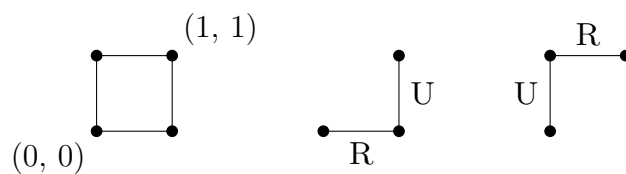
2. (20 pts) The harmonic numbers $H_0, H_1, H_2, H_3, \dots$ are defined by

$$H_0 = 0, \quad H_n = H_{n-1} + \frac{1}{n}, \quad \text{for } n > 0.$$

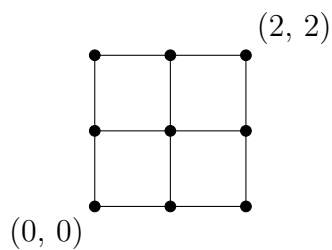
Use the principle of mathematical induction to prove that the sum of the first $n + 1$ harmonic numbers equals $(n + 1)H_n - n$, that is:

$$H_0 + H_1 + H_2 + H_3 + \cdots + H_n = (n + 1)H_n - n$$

3. (20 pts) Consider the problem of counting the “Manhattan” paths in a grid. That is starting from $(0, 0)$ move up (U) or right (R) to reach destination (n, m) , $n, m > 0$. For example there are 2 paths from $(0, 0)$ to $(1, 1)$: UR or RU:



- (a) How many paths are there from $(0, 0)$ to $(2, 2)$?



- (b) How many paths are there from $(0, 0)$ to (n, m) ?

4. (20 pts) Let T be a binary tree with n nodes and height h

(a) What is an upper bound on n in terms of the height h ?

(b) What is an lower bound on n in terms of the height h ?

(c) If T has λ leaf nodes, what is a lower bound on the height h ?

(d) If T has λ leaf nodes, is there an upper bound on the height h ?

5. (20 pts) A road map of Yoknapatawpha County is shown below. The numbers along each road indicate the mileage between towns. All roads are unpaved, but the County Commission wants to pave some roads so that there is at least one paved road between each town. The cost of paving a road is \$100,000 per mile. Determine which roads to pave so that the total cost is a minimum.

