Discrete Mathematics Comprehensive Examination (Spring 2005)

March 25, 2005

Sign the exam with your student number, not your name _____

There are eight (8) equally weighted problems in this examination. Answer any six (6) of them to the best of your ability. Be sure to read carefully and address each instruction called out in a problem and its parts. Indicate below, by circling the appropriate problem numbers, which problems you wish to be graded; otherwise, the first six will be selected for you.

1 2 3 4 5 6 7 8.

1. Consider the following statements:

   (a) If it is sunny, then I will not take my umbrella.

   (b) $(\exists x)(\forall y)[x < y \wedge 2x > y]$

   (c) $p \Longleftrightarrow q$

   i. Negate each of the above statements.

   ii. Write the negated statements from above without any negation symbols or negative words (*not*, *false*, ...).

   iii. For parts (1a) and (1c), construct a truth table to confirm your result.

2. Consider two algebraic structures $\Sigma_1 = (\{a, b, c, d\}, \clubsuit, \diamondsuit)$ and $\Sigma_2 = (\{w, x, y, z\}, \heartsuit, \spadesuit)$ whose operation tables are listed below. One of them is a Boolean algebra and the other is not (it is structure called a field).

   (a) State which one must be the Boolean algebra (because the other cannot be a Boolean algebra).

   (b) Describe criteria you used to ascertain which is the Boolean algebra and which is not.

$\clubsuit$	a	b	c	d
a	a	b	c	d
b	b	b	c	c
c	c	c	c	c
d	d	c	c	d

$\diamondsuit$	a	b	c	d
a	a	a	a	a
b	a	b	b	a
c	a	b	c	d
d	a	a	d	d

$\heartsuit$	w	x	y	z
w	w	x	y	z
x	x	w	z	y
y	y	z	w	x
z	z	y	x	w

$\spadesuit$	w	x	y	z
w	w	w	w	w
x	w	x	y	z
y	w	y	z	x
z	w	z	x	y

3. How many distinct permutations of the letters *muumuu* are there? How does this particular type of permutation problem relate to a combination problem?

4. Parse the expression

$$2 \times 3 + 4 \times 6 \div (9 - 1)$$

into a binary tree representation, assuming standard precedence for arithmetic operators. Then rewrite the expression in postfix form, corresponding to postorder traversal of the tree. Then rewrite the expression in prefix form, corresponding to preorder traversal of the tree.

5. Demonstrate, by applying the definition of asymptotic order of magnitude, that $2n^3 + 6n^2 + 6n + 2 = \Theta(n^3)$

6. Prove that $2^n < n!$ for all integers $n > 3$ by using mathematical induction

7. Show that

$$\{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$$

constitutes a partial ordering but not an equivalence relation on $\{1, 2, 3, 4\}$. Draw a Hasse diagram for this relation. Note any greatest, least, maximal, and minimal elements.

8. Fill in each of the following blanks with *always*, *sometimes*, or *never* as appropriate. (Use sometimes if and only if the statement is true for some cases and false for other cases. Read the statements carefully, and do not make any assumptions beyond what is explicitly stated.)

   (a) Two nested existential quantifiers can _____ be interchanged without changing the value of the quantified statement

   (b) A nested existential and universal quantifier can _____ be interchanged without changing the value of the quantified statement.

   (c) The converse of a true conditional is _____ true.

   (d) It is _____ valid to conclude $\neg p$ from $p \implies q$ and $\neg q$.

   (e) $\emptyset$ is _____ an element of an arbitrary set $\mathbb{S}$.

   (f) $\emptyset$ is _____ a subset of an arbitrary set S.

   (g) An injective (one-to-one) function is _____ invertible.

   (h) A graph _____ has an odd number of nodes with odd degree.

   (i) The equivalence classes in an equivalence relation _____ have the same cardinality.

   (j) If $f(n) = O(g(n))$, then it is _____ true that $g(n) = \Theta(f(n))$.