A secretary problem with two decision makers

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Abstract

n applicants of similar qualification are on an interview list and their salary demands are from a known and continuous distribution. Two managers, I and II, will interview them one at a time. Right after each interview, manager I always has the first opportunity to decide to hire the applicant or not unless she has hired one already. If manager I decides not to hire the current applicant, then manager II can decide to hire the applicant or not unless he has hired one already. If both managers fail to hire the current applicant, they interview the next applicant, but both lose the chance of hiring the current applicant. If one of the managers does hire the current one, then they proceed with interviews until the other manager also hires an applicant. The interview process continues until both managers hire an applicant each. However, at the end of the process, each manager must have hired an applicant. In this paper, we first derive the optimal strategies for them so that the probability that the one he hired demands less salary than the one hired by the other does is maximized. Then we derive an algorithm for computing manager II's winning probability when both managers play optimally. Finally we show that manager II's winning probability is strictly increasing in n, is always less than c, and converges to c as $n \to \infty$, where $c \doteq 0.3275624139\cdots$ is a solution of the equation $\ln(2) + x\ln(x) = x$.

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1 Introduction

There are n applicants of similar qualification on an interview list. Their salary demands are from a known distribution. Two managers, I and II, will interview them one at a time. Right after each interview, manager I always has the priority to decide to hire the current applicant or not unless he has hired one already. If manager I decides not to hire the current applicant, then manager II can decide to hire the current applicant or not unless he has hired one already. If both managers decide not to hire the current applicant, they will interview the next applicant, but both lose the chance of hiring the current applicant. If one of the managers does decide to hire the current one, then they will proceed with the interview until the other manager also hires an applicant. The interview process will continue until both managers hire an applicant. However, by the end of the list, each manager should have hired an applicant. This means that manager II has to hire applicant $n-1$ if both managers have not yet hired one from the first $n-2$ applicants and manager I wants to hire applicant n.

Even though their salary demands are from a known distribution, they come in in a random order. The following questions are meaningful and interesting.

1. What is the optimal strategy for manager I so that the probability that the one he hired demands less salary than the one hired by manager II does is maximized ?

2. What is the optimal strategy for manager II so that the probability that the one he hired demands less salary than the one hired by manager I does is maximized ?

3. When both managers use their optimal strategies, what is manager II's winning probability and what is the limit of this probability as $n \to \infty$?

In [1], Berry, Chen, and Rosenberg studied the setting in which both managers only know that applicants come in in a random order and their salary demands are distinct. They showed that for all $n \geq 3$ and both managers play optimally, manager II's winning probability is always less than $\frac{1}{2} - a$ for some constant a in $(0, 0.1]$, but is not a monotone function of n in this setting. In [4], Yang presented this setting as a classroom game. He derived a simple strategy for manager I. However, his simple strategy performs well but not optimal for manager I.

In this note, we consider the setting in which the distribution of applicants' salary demands is known and continuous. We first derive the optimal strategy for manager I and the optimal strategy for manager II. Then we derive an algorithm for computing manager II's winning probability when both managers play optimally. Finally we show that in this new setting, manager II's winning probability is strictly increasing in n, is always less than c, and converges to c as $n \to \infty$, where $c \doteq 0.3275624139\cdots$ is a solution of the equation $\ln(2) + x\ln(x) = x$.

For $k = 1, 2, \ldots, n$ and $n \geq 2$, let X_k denote the k^{th} applicant's salary demand. Since we assume the common distrubution is known and continuous, we can and do assume that the common distribution is the uniform distribution over the interval $(0, 1)$. Since $P(X_2 \geq X_1, X_3 \geq X_1, \ldots, X_k \geq X_1 | X_1 = x) = \prod_{i=2}^{k} P(X_i \geq x) = (1-x)^{k-1}, P(X_i < X_1$ for some $i = 2, \ldots, k | X_1 = x) = 1 - (1-x)^{k-1}$. Suppose that both managers did not hire any applicants from the previous interviews, suppose that the current applicant's salary demand is x, and suppose that there are $k-1$ more applicants to come. Then manager I should hire the current applicant if $(1-x)^{k-1} \geq \frac{1}{2}$ since his (conditional) winning probability is $(1-x)^{k-1} \geq \frac{1}{2}$. Otherwise, he should postpone his decision to the next interview. On the other hand, if manager II did hire an applicant already, manager I will hire the current applicant if his salary demand is less than the salary demand of the one hired by manager II. Otherwise, manager I should interview the next applicant, or he will be forced to hire the last applicant. Now we briefliy show that this strategy is optimal for manager I: (1) Suppose that manager I ignores manager II's action completely and applies the optimal best choice strategy described in section 3 of [3] with which he gets the very best applicant with asymptotic probability 0.58 and manager II cannot prevent this because manager I always has the priority. Hence manager I's winning probability $P > \frac{1}{2}$ if he plays optimally. (2) Suppose that the strategy described above were not optimal for manager I, then manager II would be able to hire the current applicant and win at least as likely as manager I playing optimally later on. This contradicts to the fact that $P > \frac{1}{2}$. Therefore, the optimal strategy for manager I can be described as follow.

(i) Suppose that both managers did not hire an applicant from the previous interviews, suppose that the current applicant's salary demand is x, and suppose that there are $k-1$ more applicants to come. Then manager I should hire the current applicant if $(1-x)^{k-1} \geq \frac{1}{2}$. Otherwise, he should postpone his decision to the next interview.

(ii) Suppose manager II has hired an applicant from the previous interviews, and suppose that manager I has not hired one yet. Then manager I will hire the current applicant if the current applicant's salary demand is less than the salary demand of the one hired by manager II. Otherwise, he should continue to search and to hire an applicant whose salary demand is less than the salary demand of the one hired by manager II or he will be forced to hire the last applicant.

The optimal strategy for manager II is more complicated. He needs to compute the expected (conditional) winning probability if he does not hire the current applicant and postpones his decision to the next interview. Suppose that neither manager hired an applicant from the previous interviews, suppose that there are k applicants (including the current one) available, and suppose that the current applicant's salary demand is x and $(1-x)^{k-1} < \frac{1}{2}$, then manager II should hire the current applicant if $(1-x)^{k-1} \geq q(k-1)$, where $q(k-1)$ is defined below. Otherwise, he should postpone his decision to the next interview. Therefore, the optimal strategy for manager II can be described as follow.

(i) If manager I has hired an applicant already, then manager II will continue to search an applicant whose salary demand is less than the salary demand of the one hired by manager I, or he will be forced to hire the last applicant.

(ii) Suppose that manager I did not hire an applicant, the current applicant's salary demand is x, and there are $k-1$ more applicants available in the future. Then manager II should hire the current one if $q(k-1) \leq (1-x)^{k-1} < \frac{1}{2}$. Otherwise, he should postpone his decision to the next interview.

From now on we will assume that both managers play optimally. For each x in $(0,1)$, let $q(k|x)$ denote the conditional probability that manager II will win given that there are k applicants still available for both managers (including the current one), both managers have not yet hired any applicant from the previous interviews, and the current applicant's salary demand is x. Then by definition of total probability, $q(k) = \int_0^1 q(k|x)dx$ is manager II's winning probability when there are k applicants available for both managers. First we will derive a recursive formula to compute $q(n)$ for $n = 2, 3, \ldots$. Then we will prove the main theorem of this note.

Assume that $n \geq 2$, and there are n applicants (including the current one) available, and assume that both managers have not hired any applicant from the previous interviews.

Suppose that the current applicant's salary demand is x $(0 < x < 1$, since we can and do assume the common distribution is the uniform distribution over the interval $(0,1))$. Then $q(n|x) = 1-(1-x)^{n-1}$ if $(1-x)^{(n-1)} \geq \frac{1}{2}$ (since manager I will hire the current one), $q(n|x) = (1-x)^{n-1}$ if $q(n-1) \leq (1-x)^{n-1} < \frac{1}{2}$ (since manager I does not want to hire the current one and manager II will hire the current one), and $q(n|x) = q(n-1)$ if $(1-x)^{n-1} < q(n-1)$ (since both managers will postpone their decisions to the next interview). Therefore,

$$q(n) = \int_0^1 q(n|x)dx = \frac{n-1}{n}\{1-(\frac{1}{2})^{1/(n-1)} + [q(n-1)]^{n/(n-1)}\}$$

for all $n = 3, 4, \ldots$. Here,

$$q(2) = \int_0^{1/2} x dx + \int_{1/2}^1 (1-x)dx = \frac{1}{4}.$$

Main theorem. Manager II's winning probability $q(n)$ is always less than c for all $n \geq 2$, $q(n)$ is strictly increasing in n, and $q(n) \to c$ as $n \to \infty$, where $c \doteq 0.32756\cdots$ is a solution of the equation $\ln 2 + x\ln x = x$.

Proof. First, we will show that $\frac{1}{4} \leq q(n) < q(n+1) < \frac{1}{3}$ for all $n \geq 2$. To see this, for $0 \leq x \leq \frac{1}{2}$ and $0 \leq y \leq \frac{1}{3}$, let $\phi(x,y) = 1 - 2^{-x} + y^{1+x} - (1+x)y$. For each x in $(0, \frac{1}{2}], \phi(x, \frac{1}{4}) > 0$ and $\phi(x, \frac{1}{3}) < 0$. Since for each x in $(0, \frac{1}{2}]$, $\phi(x,y)$ is a strictly decreasing function of y for $0 \leq y \leq 1$, $\phi(x, \frac{1}{4}) > 0$, and $\phi(x, \frac{1}{3}) < 0$, there exists a unique $y(x)$ in $(\frac{1}{4}, \frac{1}{3})$ such that $\phi(x, y(x)) = 0$ for each x in $(0, \frac{1}{2}]$. Hence for each $n = 2, 3, ...$, there exists a unique y_n in $(\frac{1}{4}, \frac{1}{3})$ such that $\phi(1/n, y_n) = 0$. Furthermore, $\phi(1/n, u) > 0 > \phi(1/n, \nu)$ for all $0 \leq u < y_n < \nu \leq \frac{1}{3}$ and for all $n = 2, 3, ...$ Since $q(2) = \frac{1}{4}$ and $y_2 \doteq 0.31074\cdots$, $q(2) < y_2 < \frac{1}{3}$. Now suppose that $q(k-1) < q(k)(q(1) = 0)$ and $q(k) < y_k < \frac{1}{3}$ for all $k = 2, 3, ..., n$. Then

$$\begin{aligned} q(n+1) - q(n) &= \frac{n}{n+1}\{1 - 2^{-1/n} + [q(n)]^{1+1/n} - (1+1/n)q(n)\} \\ &= \frac{n}{n+1}[\phi(\frac{1}{n}, q(n)) \\ &> 0 \end{aligned}$$

since $q(n) < y_n$, i.e., $q(n) < q(n+1)$. On the other hand

$$\begin{aligned} q(n+1) &= \frac{n}{n+1}\{1 - 2^{-1/n} + [q(n)]^{1+1/n}\} \\ &< \frac{n}{n+1}\{1 - 2^{-1/n} + (y_n)^{1+1/n}\} = y_n \end{aligned}$$

since $\frac{1}{4} \leq q(n) < y_n$. If we can show that $y_n < y_{n+1}$, then by mathematical induction, we have $q(n) < y_n < \frac{1}{3}$ and $q(n) < q(n+1)$ for all $n = 2, 3, ...$

Table 1 below reveals that $y_2 < y_3 < \cdots < y_{25}$. We need only to show that $y_n < y_{n+1}$ for all $n \geq 25$. To see this, we will show that $y(x)$ is strictly decreasing in x for all x in $(0, 0.04]$. Since $y_2 > 0.3$, we will restrict our study of $\phi(x,y)$ for $0 < x \leq 0.04$ and $0.3 \leq y \leq \frac{1}{3}$. Notice that $\phi(x,y) = 0$ if and only if $h(x,y) = \phi(x,y)/x = 0$ since $x > 0$. Now

$$\begin{aligned} h(x,y) &= \{1 - 2^{-x} + y^{1+x} - (1+x)y\}/x \\ &= \ln 2 + y \ln y - y + \sum_{k=2}^{\infty} \{[y(\ln y)^k - (-\ln 2)^k] x^{k-1}/k!\}. \end{aligned}$$

Notice that for $0.3 \leq y \leq \frac{1}{3}$, $\frac{1}{2}[y(\ln y)^2 - (\ln 2)^2] \leq \frac{1}{2}[0.3(\ln 0.3)^2 - (\ln 2)^2] < -0.02$ and $\sum_{k=3}^{\infty}[y|\ln y|^k + (\ln 2)^k]/k! < 0.2$. Now if $0 < x < x_1 \leq 0.04$, $0.3 \leq y \leq \frac{1}{3}$, and $h(x_1, y) = 0$, then

$$\begin{aligned} h(x,y) &= h(x,y) - h(x_1,y) \\ &= (x - x_1)\left\{\frac{1}{2}[y(\ln y)^2 - (\ln 2)^2]\right. \\ &\qquad \left. + \sum_{k=3}^{\infty} \{[y(\ln y)^k - (-\ln 2)^k][x^{k-2} + x^{k-3}x_1 + \ldots + x x_1^{k-3} + x_1^{k-2}]\}/k!\right\} \\ &> (x - x_1)\left[-0.02 + 0.2 \sum_{k=3}^{\infty} (k-1)(0.04)^{k-2}\right] > 0. \end{aligned}$$

Since $h(x,y)$ is strictly decreasing in y (for a fixed x) and $h(x,y) > 0$, $y(x) > y$. Hence $y(x)$ is strictly decreasing in x for x in $(0, 0.04]$ and $0 < y(x) < \frac{1}{3}$ (since $h(x, \frac{1}{3}) < 0$). Therefore, $y_n < y_{n+1}$, $q(n+1) < y_n < y_{n+1}$, and $q(n) < q(n+1) < \frac{1}{3}$ for all $n \geq 2$.

Next we are going to show that $q(n) < c \doteq 0.32756\cdots$, where c is a solution of the equation $\ln 2 + x \ln x - x = 0$. Notice that if $c < d < \frac{1}{2}$, then $\ln 2 + d \ln d - d < 0$. Since $q(2) = \frac{1}{4} < d$, it is sufficient to show that $q(n+1) < d$ if $q(n) < d$. Now suppose that $q(n) < d$, then

$$\begin{aligned} q(n+1) &= \frac{n}{n+1}\{1 - 2^{-1/n} + [q(n)]^{1+1/n}\} \\ &< \frac{n}{n+1}\{1 - 2^{-1/n} + d^{1+1/n}\}. \end{aligned}$$

It is sufficient to show that $1 - 2^{-1/n} + d^{1+1/n} - d(1 + 1/n) < 0$. For each x in $[0, \frac{1}{2}]$, let $k(x) = 1 - 2^{-x} + d^{1+x} - d(1+x)$. Notice that $k(0) = 0$, $k'(x) = 2^{-x}\ln 2 + d^{1+x}\ln d - d$, and $k''(x) = -2^{-x}(\ln 2)^2 + d^{1+x}(\ln d)^2$. Since $d(\ln d)^2$ is strictly decreasing for $e^{-2} \leq d \leq 1$, $k''(x) \leq (-2^{-x} + d^x)(\ln 2)^2 < 0$ for $\frac{1}{4} \leq d < \frac{1}{2}$ and $0 < x \leq \frac{1}{2}$. Hence $k'(x)$ is strictly decreasing for $0 \leq x \leq \frac{1}{2}$. Notice that $k'(0) = \ln 2 + d \ln d - d < 0$ since the function $\ln 2 + z \ln z - z$ is strictly decreasing for $0 < z < 1$ and $d > c$, where $0.3 < c < \frac{1}{3}$ and $\ln 2 + c \ln c - c = 0$. Therefore, $k'(x) < k'(0) < 0$, i.e., $k(x)$ is strictly decreasing in x and $k(x) < k(0) = 0$ for all $0 < x \leq \frac{1}{2}$. Hence

$$1 - 2^{-1/n} + d^{1+1/n} - d(1 + 1/n) < 0,$$

i.e., $q(n+1) < d$. Therefore, $q(n) \leq c$ for all $n \geq 2$. Since $q(n) < q(n+1)$ for all $n \geq 2$. If $q(n) = c$, then $q(n+1) > c$. Therefore, $q(n) < c$ for all $n \geq 2$.

Finally we are in a position to show that $q(n) \to c$ as $n \to \infty$. Since $q(n)$ is strictly increasing and bounded above by c, $q(n) \to t \leq c$ as $n \to \infty$ where t is a constant. If $t < c$, then $\varepsilon = c - t$ for some constant $\varepsilon > 0$. Since $q(n) > q(6) > 0.3$ for all $n \geq 7$, we assume that $0.3 < t < c$. Since $0.3 < t < c$, $\ln 2 + t \ln t > t$. Now let $\phi(x, y) = 1 - 2^{-x} + y^{1+x} - (1+x)y$ for all $0 \leq x \leq \frac{1}{2}$ and $\frac{1}{4} \leq y \leq t$. Then $\phi_x = 2^{-x} \ln 2 + y^{1+x} \ln y - y$. Since $y^{1+x} \ln y - y$ is strictly decreasing in y if $0 < y < 1$ and $x > 0$, $\phi_x \geq 2^{-x} \ln 2 + t^{1+x} \ln t - t$ if $\frac{1}{4} \leq y \leq t$. Since $\ln 2 + t \ln t - t > 0$, $2^{-x} \ln 2 + t^{1+x} \ln t - t \geq \Delta > 0$ if $0 \leq x \leq \delta$, where $0 < \delta \leq \frac{1}{2}$. Therefore, $\phi_x \geq \Delta$ for all $0 < x \leq \delta$ and $\frac{1}{4} \leq y \leq t$. Notice that

$$h(\frac{1}{n}, q(n)) = 1 - 2^{-1/n} + [q(n)]^{1+1/n} - \left(1 + \frac{1}{n}\right) q(n) \geq h(0, q(n)) + \frac{1}{n}\Delta$$

if $1/n \leq \delta$. Hence $q(n+1) - q(n) \geq (1/(n+1))\Delta$ and $q(n+k) - q(n) \geq \sum_{j=1}^{k} (1/(n+j))\Delta$ for all $k = 1, 2, \ldots$ if $1/n \leq \delta$, which implies $q(n+k) \to \infty$ as $k \to \infty$ and we get a contradiction. Therefore, $q(n) \to c$ as $n \to \infty$. The proof of the main theorem now is complete.

Table 1 gives Manager II's winning probability $q(n)$ for various n

Remark. It is interesting to observe that $q(n)$ will converge to c as $n \to \infty$ for any initial value $q(2) = d$ if d is in the interval $[0, 1]$. From the proof of the main theorem, $q(n) \leq q(n+1)$ if $q(n) \leq y_n < y_{n+1}$ and $q(n) > q(n+1)$ if $y_n < q(n)$. Since $y_2 < y_3 < \cdots$ and $\lim_{n\to\infty} y_n = c$, $q(n) \to c$ if $q(n) \leq y_n$ for some n. Now if $q(n) > y_n$ for all $n \geq 2$, then, by a similar argument as the one used in the proof of the main theorem, we can again show that $q(n) \to c$ as $n \to \infty$. We omit the details.

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Note added in proof

Immediately prior to publication, the authors were informed by the referee of a paper by Enns and Ferenstein, 'The Horse Game', (*J. Operat. Res. Soc. Japan,*, **28**, 1985, pp. 51–62) which also contains this problem. We apologize to those authors for our oversight in not finding and referencing their paper, which gives a treatment of the problem with horses rather than secretaries. However, in their paper the sequence u_n is claimed monotonic and bounded, but this claim is not proved. We call this sequence $q(n)$ and prove with great difficulty these properties with the introduction of the auxiliary function $h(x, y)$. We consider the justification of the sequence's properties a major step in both papers' conclusions.

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Table 1
Manager II's winning probability $q(n)$

n	$q(n)$	n	$q(n)$
2	0.250000	30	0.322067
3	0.278595	40	0.323340
4	0.291191	50	0.324121
5	0.298408	60	0.324651
6	0.303124	70	0.325036
7	0.306465	80	0.325328
8	0.308964	90	0.325558
9	0.310908	100	0.325744
10	0.312467	200	0.326606
11	0.313746	300	0.326907
12	0.314815	400	0.327061
13	0.315724	500	0.327156
14	0.316506	600	0.327220
15	0.317186	700	0.327266
16	0.317784	800	0.327301
17	0.318314	900	0.327329
18	0.318788	$1,000$	0.327351
19	0.319212	$2,000$	0.327453
20	0.319595	$3,000$	0.327488
21	0.319944	$4,000$	0.327506
22	0.320262	$5,000$	0.327517
23	0.320553	$10,000$	0.327539
24	0.320821	$20,000$	0.327550
25	0.321068	$50,000$	0.327557