Archimedes Dream, Notes

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Elementary notes

Assume a circle of unit radius, that is, of area π.

1. A square circumscribed, that is, tightly drawn outside the circle so that the middle of each side touch the circle, will have side length 2, and therefore area 4.

2. A square inscribed, that is, tightly drawn inside the circle so that its four corners touch the circle, has area 2. This can be seen by adjusting the inscribed square to the circumscribed square so that the inscribed square is the adjoining of the midpoints of the circumscribed square's edges. Consider the outside square as a piece of paper. The inside square is realized by folding the outside square's four corners to the middle. This means that the area of the outside square is enough to cover the inside square twice. From area 2, we conclude the inside square has side length $\sqrt{2}$.

3. An inscribed regular octagon is realized from an inscribed square by pushing out from the mid-point of each side to touch the circle. This adds to the area of the square 4 triangles, of base the square's width, $\sqrt{2}$. The height of the triangle is calculated by drawing a radial line from the center of the circle to the new point of contact. The height is the length of this radial line, 1, minus the distance to the triangle base, which is half the width of the square, $\sqrt{2}/2$. The area of one triangle is therefore,

   $$(1/2)\sqrt{2}(1-\sqrt{2}/2) = (1/2)(\sqrt{2}-1)$$

   and of all four triangles, plus the square's area of 2,

   $$4(1/2)(\sqrt{2}-1)+2 = 2\sqrt{2}-2+2 = 2\sqrt{2}.$$

4. A circumscribed regular octagon is realized from a circumscribed square by cutting off four triangles from the points of the square. Drawing a radial line from the center of the circle to the corner of the outside square, the height of a cut triangle is the length of this line minus the distance to where the line cuts the circle. The length of this line is one half the diagonal of a 2 by 2 square, $\sqrt{2}$, and the length to the circle is 1, so the triangle's height is $\sqrt{2}-1$.

To calculate the base, one must recognize that this triangle has two equal angles, and hence these angles are 45 degrees. The radial line bisects the 90 degree angle of the square, making a 45 degree angle, hence the height cuts the triangle into two smaller isoceles triangles. This gives that the base is twice the height, $2(\sqrt{2}-1)$. The area of one triangle is therefore,

$$(1/2)(\sqrt{2}-1)2(\sqrt{2}-1) = 2 - 2\sqrt{2} + 1 = 3 - 2\sqrt{2}$$

and subtracting four such triangles from the original area of 4,

$$4 - 4(3 - 2\sqrt{2}) = 4 - 12 + 8\sqrt{2} = 8\sqrt{2} - 8 = 8(\sqrt{2} - 1).$$

This completes the heavy math for Archimedes Dream's parts 1 and 2.

We can continue with continued fraction expansions of $\sqrt{2}$ to get bounds as rational numbers, I would not like to use decimal approximations since this is not beautiful.

Further explorations

How so we continue from 8 to 16, 16 to 32, and so on, to regular 2^k-gon's, circumscribed and inscribed, which give an infinite sequence of ever improving upper and lower bounds to *pi*? To this end I would like to quickly jot down the following observations and proofs, before I throw out the scrap paper. I acknowledge G. M. Phillips, *Archimedes the numerical analyst,* Amer. Math. Monthly, 88(3), March 1981, reprinted in *Pi: A source book,* Lennart Berggren, Jonathan Borwein and Peter Borwein, Springer 1997.

Let A_n by the area of a regular circumscribed n-gon, a_n the same for the inscribed n-gon, P_n be the semi-perimeter of a regular circumscribed n-gon, and p_n the same for the inscribed n-gon.

Lemma 1 *A regular n-gon circumscribed around a unit circle has area equal to its semi-perimeter, $P_n = A_n$.*

Proof: Consider a single triangular sector of the circumscribed n-gon. It has height 1 and base x. Therefore both the area and the semi-perimeter are $nx/2$.

Lemma 2 *A regular n-gon inscribed in a unit circle has semi-permeter equal to the area of a regular inscribed $2n$-gon, $p_n = a_{2n}$.*

Proof: Consider a single triangular sector of the inscribed n-gon. It has height 1 and base x. Therefore the semi-permeter is $p_n = nx/2$. Consider the two triangular sectors of the $2n$-gon which fit neatly inside this one sector of the n-gon. They are two triangles back to back, of common base 1 and each of height $x/2$. This gives area $a_{2n} = nx/2 = p_n$.

I consider this very interesting. Using trigonometric functions, we can quickly get formulae for a_n and such, and show the simple relationships between these quantities used by Archimedes.

Lemma 3 $a_n = n \sin(\pi/n) \cos(\pi/n)$.

Proof: Consider a single triangular sector of the inscribed n-gon, and bisect with a radial line from the circle's center. This forms two right triangles with hypotenuse 1 and angle is π/n, so $\sin(\pi/n)$ and $\cos(\pi/n)$ are the height and base, respectively. Add up the area of these $2n$ triangles.

Lemma 4 $A_n = n \tan(\pi/n)$.

Proof: Consider a single triangular sector of the circumscribed n-gon, and bisect with a radial line from the circle's center. This forms two right triangles of base 1 and angle π/n. Hence $\tan(\pi/n)$ is the triangle's height. Add up the area of these $2n$ triangles.

Lemma 5 *The area of an regular inscribed $2n$-gon is the geometric mean of the areas of regular inscribed and circumscribed n-gons, $a_{2n} = \sqrt{a_n A_n}$.*

Proof: Verify the identity,

$$\sqrt{a_n A_n} = \sqrt{n \sin(\pi/n) \cos(\pi/n) n \tan(\pi/n)} = n \sin(\pi/n).$$

Now apply the double angle formula,

$$n \sin(2\pi/(2n)) = n \sin(\pi/(2n)) \cos(\pi/(2n)) = a_{2n}.$$

Lemma 6 *The semi-perimeter of a regular circumscribed $2n$-gon is half the harmonic mean of the semi-perimeters of regular inscribed and circumscrbied n-gons,*

$$1/P_{2n} = (1/2)(1/p_n + 1/P_n).$$

The same is true of perimeters.

Proof: Given the relationships between semi-perimeters and areas, we show,

$$1/A_{2n} = (1/2)(1/a_{2n} + 1/A_n).$$

Using a form of the half-angle formula for tangent,

$$\tan\theta = \pm \sin(2\theta)/(1 + \cos(2\theta))$$

we have,

$$1/A_{2n} = \frac{1}{2n \tan(\pi/(2n))} = \pm \left(\frac{1}{2n \sin(\pi/n)} + \frac{1}{2n \tan(\pi/n)} \right) = (1/2)(1/a_{2n} + 1/A_n).$$

To get the statement for perimeters, multiply both sides by 1/2 and rename variables.

As stated, there are some very nice symmetries in these formulas, adding to the curious interrelationships of these quantities. Other forms of the these equations are:

Corollary 1 $1/A_{2n} = (1/2)(1/a_{2n} + 1/A_n)$ *and* $p_{2n} = \sqrt{P_{2n} p_n}$.

As a note, if one attempts to construct relationships working with the other version of the half-angle formula,

$$\tan\theta = \pm(1 - \cos 2\theta)/\sin 2\theta$$

This gives,

Lemma 7 $A_{2n}/(2n^2) = 1/A_n - 1/a_{2n}$.

Note that this formula is numerically unstable due to a subtraction and then a large amplification of the truncated result. Not only did Archimedes derive his formulas without aid of a developed theory of trigonometry, but also he found numerically well-conditioned recurrence relations and avoided the badly behaved relations.

Here are some values for a_n and A_n,

n	a_n	A_n	a_n/A_n
4	2	4	1/2
8	$2\sqrt{2}$	$8(\sqrt{2}-1)$	$(2+\sqrt{2})/4$
16	$4\sqrt{2-\sqrt{2}}$	$8\sqrt{2}(2-\sqrt{2})/(2+\sqrt{2+\sqrt{2}})$	

Figure 1: Values for $n = 2^i$

There is another approach to recurrence, which might be easier for hand calculation. Here is a simple consequence of our other formulas.

Corollary 2 $a_n/A_n = \cos^2 \pi/n$.

Lemma 8

$$\frac{a_{2n}}{A_{2n}} = \frac{\sqrt{a_n/A_n} + 1}{2}$$

Proof: Follows from half-angle formula,

$$\cos^2(\theta/2) = (\cos\theta + 1)/2$$

and the previous corollary.

The result of this formula is the following sequence for a_n/A_n:

$$\frac{2}{4}, \frac{2+\sqrt{2}}{4}, \frac{2+\sqrt{2+\sqrt{2}}}{4}, \frac{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}{4}, \ldots$$

Finally, it is easy enough to derive,

$$a_{2n} = a_n \sqrt{A_n / a_n}$$

giving the sequence for a_n,

$$2, \frac{4}{\sqrt{2}}, \frac{8}{\sqrt{2}\sqrt{2+\sqrt{2}}}, \frac{16}{\sqrt{2}\sqrt{2+\sqrt{2}}\sqrt{2+\sqrt{2+\sqrt{2}}}}, \ldots$$

Note that now we can continue with continued fractions or with a geometric construction (straight-edge and compass) to construct convergents to π.

Beging with the special property of a hexagon, that it is six equalateral triangles, and if inscribed in a unit circle then the side length is one, we have the follow tabled derived from our recurrences or by direct reasoning. The direct reasoning begins with the hexagon and, to get an inscribed triangle, connects alternate vertices. The side length is twice the height of a side length one equilateral triangle. Further, the circumscribed triangle is seen to be equilateral with side length twice that of the inscribed triangle. To go the 12-gon, subdivide the 6-sector to get two triangles of height one and base 1/2. Immediately it follows that the area is one.

n	a_n	A_n	p_n	P_n
3	$3\sqrt{3}/4$	$3\sqrt{3}$	$3\sqrt{3}/2$	$3\sqrt{3}$
6	$3\sqrt{3}/2$		3	
12	3			

Figure 2: Values for $n = 3(2^i)$

References

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