Binomial Models of Russian Option

Robert W. Chen (Department of Mathematics)
Burton J. Rosenberg (Department of Computer Science)
Alan Zame (Department of Mathematics)
University of Miami
Coral Gables, Florida 33124

Abstract

The well-known Russian lookback option offers the buyer reduced regret in paying the maximum price of the asset (discounted in time) at any execution time selected by the buyer. In the continuous case an exact formula is known for the fair price of the option when there is no limit on the expiration time (pertpetual case). In the more realistic case when the expiration time is fixed at N, an exact formula is impossible but we give an elegant algorithm to compute the fair price and the optimal execution boundary in order N computations. This is possible only because we prove that the optimal execution boundary must have a specific form which is exploited in our algorithm. We also give a fast algorithm to compute the fair prices of the option at all possible nodes in order N^2 computations.

As in [1], we consider a N-period binomial model of a security market, consisting of two assets: a "risk-free" savings account $\{B_n | 0 \leq n \leq N\}$ and a "risky" security $\{S_n | 0 \leq n \leq N\}$ whose price processes are given by the following recurrence relations:

$$B_n = (1+r)B_{n-1}, 1 \leq n \leq N, B_0 = 1,$$

$$S_n = (1+\rho)S_{n-1}, 1 \leq n \leq N, S_0 > 0,$$

where $r > 0$ is the risk free interest rate and $1+\rho = u$ or $\frac{1}{u}$ for some constant $u > 1+r$.

By [6], a "Russian option" (or a discounted look back option of American type) can be viewed as a contract between two parties, a buyer and a seller. Specified as data is the risky security price process $\{S_n | n = 0, 1, 2, ..., N\}$. If they make an agreement at time n with price history $\{S_i | i = 0, 1, 2, ..., n\}$, the buyer pays the seller an amount Z_n equal to the option fair price at time n with the price history $\{S_i | i = 0, 1, 2, ..., n\}$. The buyer then has the right to exercise his option at any time τ, where $n \leq \tau \leq N$. If the option is exercised at time τ, then the seller pays $\beta^\tau Y_\tau$ to the buyer, where $Y_\tau = max\{S_0, S_1, S_2, ..., S_\tau\}$ is the maximum price of the risky security up to time τ and β $(0 < \beta \leq 1)$ is the discounting factor agreed between the buyer and seller.

Without loss of generality, we can and do assume $S_0 = 1$. Since $S_n = uS_{n-1}$ or S_{n-1}/u for all $n = 1, 2, ..., N$, $S_n = u^j$ and $Y_n = u^k$, where $0 \leq k \leq n$, $j \leq k$, and $j = 2k - n + 2i$ for some non-negative integer i. For each price history $\{S_0, S_1, S_2, ..., S_n\}$, there is a node (a triple) (n, j, k) associated with this price history, where $S_n = u^j$ and $Y_n = u^k$. The option price at time n with price history $\{S_0, S_1, S_2, ..., S_n\}$ is a function of n, j, k. Let $E(n, j, k)$ denote the option price at node (n, j, k). For the ease of presentation, we will call a node (n, j, k) accessible if $0 \leq k \leq n$, $j \leq k$, and $j = 2k - n + 2i$ for some non-negative integer i. It is easy to see that there are $O(n^2)$ accessible nodes for each time n and there are $O(N^3)$ accessible nodes in total. If we just use a backward induction procedure, it will take $O(N^3)$ computations to compute all option prices at all accessible nodes since we need them to determine the optimal execution boundary and the optimal execution time for the buyer. In this paper, we prove some theorems showing that the optimal execution boundary for the buyer of a Russian option is of a monotonic form and bounded, and the option fair price at time 0 depends only on the specific values on and below this optimal execution boundary. By a backward induction procedure on these specific values, we can determine the optimal execution boundary and the option price at time 0 in order N computations. We can also compute the option prices at all $O(N^3)$ accessible nodes in order N^2 computations. This is possible only because we can prove that for each time n, even though there are $O(n^2)$ possible nodes, only $O(n)$ option prices are relevant.

If the buyer has not exercised his option before the expiration time N, then he certainly should exercise his option at time N and receives $\beta^N u^k$, where $S_N = u^j$ and $Y_N = u^k$. Therefore, $E(N, j, k) = \beta^N u^k$ if (N, j, k) is accessible. For any time $n < N$, if the buyer has not exercised his option before time n, then he will get $\beta^n u^k$ if he exercises his option now, where $S_n = u^j$, $Y_n = u^k$, and (n, j, k) is accessible. If he still does not want to exercise his option now, then the expected value of his option at time $n+1$ will be

$$pE(n+1, j+1, max(j+1, k)) + (1-p)E(n+1, j-1, k),$$

where $p = \{u(1+r)-1\}/\{u^2-1\}$. Hence the value of the expected value of his option at time n is

$$\alpha\{pE(n+1, j+1, max(j+1, k)) + (1-p)E(n+1, j-1, k)\},$$

here $\alpha = (1+r)^{-1}$. Therefore, for any accessible node (n, j, k),

$$E(n, j, k) =$$

$$max\{\beta^n u^k, \alpha\{pE(n+1, j+1, max(j+1, k)) + (1-p)E(n+1, j-1, k)\}\}.$$

$E(N-1, k, k) = \beta^{N-1} u^k$ if $\beta \leq \{(u+1)(1+r)\}/\{(2+r)u\}$ and $(N-1, k, k)$ is accessible. By mathematical induction we can show that $E(n, j, k) = \beta^n u^k$ for all accessible nodes (n, j, k), i.e., the buyer will either not buy the contract or will buy the contract and then exercise his option right away. Therefore, we will assume in this paper that $\{(u+1)(1+r)\}/\{(2+r)u\} < \beta \leq 1$.

The following lemmas are useful for identifying the optimal execution boundary and the optimal execution time for the buyer. They are exploited in our algorithm which requires only $O(N)$ computations to compute the option price at time 0, the optimal execution boundary, and the optimal execution time for the buyer. They are also exploited in another algorithm which requires only $O(N^2)$ computations to compute the option prices $E(n, j, k)$ for all $O(N^3)$ accessible nodes. Lemmas 1 and 2 can be easily proved by mathematical induction. For the ease of reference, we include them.

Lemma 1. For fixed n and k, $E(n, j, k) \leq E(n, j+2, k)$ if (n, j, k) and $(n, j+2, k)$ are accessible. Similarly, for fixed n and j, $E(n, j, k) \leq E(n, j, k+1)$ if (n, j, k) and $(n, j, k+1)$ are accessible.

Lemma 2. For all $n = 0, 1, 2, ..., N-1$ and $k = 0, 1, 2, ..., n$, $E(n, k, k) > \beta^n u^k$ if (n, k, k) is accessible.

Lemma 2 means that before the expiration time the buyer should never exercise his option at a situation in which the current security price is equal to the maximum security price up to now.

Lemma 3. For all $n = 0, 1, 2, ..., N$ and $k = 0, 1, 2, ..., n-2$, $E(n, j+2, k+2) = u^2 E(n, j, k)$ if $(n, j, k,)$ and $(n, j+2, k+2)$ are accessible.

Proof: Suppose that $n = N, j \leq k$, and $k \leq N-2$, then it is easy to see that Lemma 3 holds. Assuming that Lemma 3 holds for $n \geq m+1$ and suppose that $n = m$. Then, by Lemma 2,

$$u^2 E(m, k, k) = u^2\{\alpha[pE(m+1, k+1, k+1) + qE(m+1, k-1, k)]\}$$
$$= \alpha\{pu^2 E(m+1, k+1, k+1) + qu^2 E(m+1, k-1, k)\}$$
$$= \alpha\{pE(m+1, k+3, k+3) + qE(m+1, k+1, k+2)\} = E(m, k+2, k+2).$$

So Lemma 3 holds for $n = m, k \leq n-2$, and $j = k$. Now suppose that $n = m, k \leq n-2$, and $j < k$, then

$$u^2 E(m, j, k) = u^2 max\{\beta^m u^k, \alpha[pE(m+1, j+1, k) + qE(m+1, j-1, k)]\}$$
$$= max\{\beta^m u^{k+2}, \alpha[pu^2 E(m+1, j+1, k) + qu^2 E(m+1, j-1, k)]\}$$
$$= max\{\beta^m u^{k+2}, \alpha[pE(m+1, j+3, k+2) + qE(m+1, j+1, k+2)]\} = E(m, j+2, k+2).$$

By mathematical induction, Lemma 3 holds.

Lemma 4. For $n = 1, 2, ..., N$ and $k = 1, 2, ..., n$, $E(n, j, k) \geq uE(n, j-2, k-1)$ if (n, j, k) is accessible.

Proof: It is easy to check that Lemma 4 holds if $n = N$. Suppose that Lemma 4 holds for $n = m+1, m+2, ..., N, k = 1, 2, ..., n$, and accessible (n, j, k). Now suppose that $n = m, 1 \leq k \leq m$, and (m, j, k) accessible. If $j < k$, then

$$uE(m, j-2, k-1) = u\{max\{\beta^m u^{k-1}, \alpha[pE(m+1, j-1, k-1) + qE(m+1, j-3, k-1)]\}\}$$
$$= max\{\beta^m u^k, \alpha[puE(m+1, j-1, k-1) + quE(m+1, j-3, k-1)]\}$$
$$\leq max\{\beta^m u^k, \alpha[pE(m+1, j+1, k) + qE(m+1, j-1, k)]\} = E(m, j, k).$$

If $j = k$, then

$$uE(m, k-2, k-1) = u\{max\{\beta^m u^{k-1}, \alpha[pE(m+1, k-1, k-1) + qE(m+1, k-3, k-1)]\}\}$$
$$= max\{\beta^m u^k, \alpha[puE(m+1, k-1, k-1) + quE(m+1, k-3, k-1)]\}$$

$$\leq max\{\beta^m u^k, \alpha[pu^2E(m+1,k-1,k-1)+quE(m+1,k-3,k-1)]\}$$

$$\leq max\{\beta^m u^k, \alpha[pE(m+1,k+1,k+1)+qE(m+1,k-1,k)]\} = E(m,k,k)$$

since $u > 1$, $E(m+1,k-1,k-1) > 0$, $u^2E(m+1,k-1,k-1) = E(m+1,k+1,k+1)$, and $uE(m+1,k-3,k-1) \leq E(m+1,k-1,k)$. Therefore, Lemma 4 holds for $n = m, k = 1,2,...,m$, and (n,j,k) accessible. By mathematical induction, Lemma 4 holds.

Lemma 5. For $n = 0,1,2,...,N-1$, $E(n+1,j+1,k+1) \leq \beta uE(n,j,k)$ if (n,j,k) is accessible.

Proof: If $n = N-1$, then

$$\beta uE(N-1,j,k) \geq \beta u(\beta^{N-1}u^k) = \beta^N u^{k+1} = E(N,j+1,k+1).$$

Suppose that Lemma 5 holds for $n = m+1, m+2, ..., N-1, k = 0,1,2,...,n$, and (n,j,k) accessible. Now suppose that $n = m$. If $j = k$, then by Lemma 2,

$$\beta uE(m,k,k) = \beta u\{\alpha pE(m+1,k+1,k+1) + \alpha qE(m+1,k-1,k)\}$$

$$= \alpha p[\beta uE(m+1,k+1,k+1)] + \alpha q[\beta uE(m+1,k-1,k)]$$

$$\geq \alpha pE(m+2,k+2,k+2) + \alpha qE(m+2,k,k+1) = E(m+1,k+1,k+1).$$

If $j < k$, then

$$\beta uE(m,j,k) = \beta umax\{\beta^m u^k, \alpha[pE(m+1,j+1,k)+qE(m+1,j-1,k)]\}$$

$$= max\{\beta^{m+1}u^{k+1}, \alpha[p\beta uE(m+1,j+1,k)+q\beta uE(m+1,j-1,k)]\}$$

$$\geq max\{\beta^{m+1}u^{k+1}, \alpha[pE(m+2,j+2,k+1)+qE(m+2,j,k+1)]\} = E(m+1,j+1,k+1).$$

Therefore, Lemma 5 holds for $n = m$. By mathematical induction, Lemma 5 holds.

Theorem 1. Suppose that (n,j,k) and (n,j',k') are accessible. Then $E(n,j,k) = \beta^n u^k$ implies that $E(n,j',k') = \beta^n u^{k'}$ if $k - j \leq k' - j'$.

Proof: By Lemma 3, we can and do assume that $k = 0$ or 1 and $k' = 0$ or 1. If $k = k'$ and $k - j \leq k' - j'$, then $j' \leq j$. By Lemma 1, $E(n,j',k') = E(n,j',k) \leq E(n,j,k)$. Therefore, Theorem 1 holds. If $k = 1$ and $k' = 0$, then $j' \leq j - 1$. Since $(n,j,1)$ and $(n,j',0)$ are accessible, $j = 2 - n + 2i$ for some non-negative integer i and $j' = 0 - n + 2i'$ for some non-negative integer i'. Since $j' \leq j - 1$, $2i' \leq 1 + 2i$. Since i and i' are non-negative integers, $i' \leq i$ and $j' \leq j - 2$. By Lemma 1, we can assume that $j' = j - 2$. By Lemma 4, $E(n,j',0) = E(n,j-2,0) \leq E(n,j,1)/u$. Therefore, Theorem 1 holds. If $k = 0$ and $k' = 1$, then $j' \leq j + 1$. Since $(n,j,0)$ and $(n,j',1)$

are accessible, $j = 0 - n + 2i$ for some non-negative integer i and $j' = 2 - n + 2i'$ for some non-negative integer i'. Since $j' \leq j + 1$, $2i' \leq 2i - 1$. Since i' and i are non-negative integers, $i' \leq i - 1$ and $j' \leq 2 - n + 2(i - 1) = j$. By Lemmas 1 and 4, $E(n, j', 1) \leq E(n, j, 1) \leq E(n, j + 2, 2)/u$. By Lemma 2, $E(n, j + 2, 2) = u^2E(n, j, 0)$. Hence $E(n, j', 1) \leq uE(n, j, 0)$. Therefore, Theorem 1 holds.

For each expiration time N, let $n_N = min\{n|1 \leq n \leq N, E(n, j, k) = \beta^n u^k$ for some accessible node $(n, j, k)\}$. Since $E(N, j, k) = \beta^N u^k$ for any accessible node (N, j, k), n_N is well-defined. For each $n = n_N, n_N + 1, ..., N - 1$, let t_n be the smallest positive integer such that $E(n, j, k) = \beta^n u^k$ if (n, j, k) is accessible and $k - j \geq t_n$. By Lemma 2 and Theorem 1, t_n is well defined for all $n = n_N, n_N + 1, ..., N - 1$. For the ease of presentation, we will let $t_N = 0$ and $t_n = t_{n_N}$ for all $n = 0, 1, 2, ..., n_N - 1$.

Theorem 2. For $n = 0, 1, ..., N-1$, t_n is decreasing in n, $t_{N-1} = 1$, and $t_n \leq t_{n+1}+1$.

Proof: Suppose that $n = n_N, , n_N + 1, ..., N - 2$, (n, j, k) is accessible, and $E(n, j, k)$ $= \beta^n u^k$, by Lemma 5, $E(n + 1, j + 1, k + 1) \leq \beta u E(n, j, k) = \beta^{n+1}u^{k+1}$. Since $E(n + 1, j + 1, k + 1) \geq \beta^{n+1}u^{k+1}$, $E(n + 1, j + 1, k + 1) = \beta^{n+1}u^{k+1}$. By Theorem 1, we can conclude that $t_{n+1} \leq t_n$ for all $n = n_N, n_N + 1, ..., N - 1$. It is also easy to check that $t_{N-1} = 1$.

To show that $t_n \leq t_{n+1}+1$, by Lemma 4, it suffices to show that if $\beta^n u^k = E(n, j, k)$ and $\beta^n u^{k-1} < E(n, j, k - 1)$, then $E(n + 1, j + 1, k - 1) > \beta^{n+1}u^{k-1}$ if (n, j, k) is accessible. If $E(n+1, j+1, k-1) = \beta^{n+1}u^{k-1}$, then by Lemma 1, $E(n+1, j-1, k-1) = \beta^{n+1}u^{k-1}$. If $j \leq k - 2$, then $\alpha[pE(n + 1, j + 1, k - 1) + qE(n + 1, j - 1, k - 1)] = \alpha\beta^{n+1}u^{k-1} < \beta^n u^{k-1}$ since $0 < \alpha < 1$ and $0 < \beta \leq 1$. Hence $E(n, j, k - 1) = max\{\beta^n u^{k-1}, \alpha[pE(n+1, j+1, k-1) + qE(n+1, j-1, k-1)]\} = \beta^n u^{k-1}$ and we get a contradiction. Therefore, $t_n \leq t_{n+1}+1$ for all $n = n_N, n_N+1, ..., N-1$. Since $t_n = t_{n_N}$ for all $n = 0, 1, 2, ..., n_N - 1$ and $t_N = 0$, Theorem 2 holds for $n = 0, 1, ..., N - 1$.

For all $n = 0, 1, 2, ..., N$, let $X_n = Y_n/S_n$ and $\tau_{N,n}$ be the optimal execution time for the buyer if he bought the option at time n.

Theorem 3. For each security price process $\{S_0, S_1, S_2, ..., S_N\}$ and each $n = 0, 1, 2, ..., N$, the optimal execution time $\tau_{N,n}$ is given by $\tau_{N,n}(S_0, S_1, ..., S_N) = min\{m|m \geq n, X_m \geq u^{t_m}\}$. Moreover, the optimal execution boundary is $\{u^{t_0}, u^{t_1}, u^{t_2}, ..., u^{t_{N-1}}, u^{t_N}\}$.

In [5], Peskir studied this problem in a continuous time setting. He (Theorem 3.1 of [5]) showed that the optimal execution boundary is the unique continuous decreasing solution of a nonlinear integral equation. The results of Theorems 2 and 3 here are consistent with his result in a discrete time setting.

In [4], Kramkov and Shiryaev studied this problem in a different approach which can be stated as follows: For each $n = 1, 2, ..., N$, ϵ_n is a random variable and takes two values $+1, -1$. So $S_n = S_0 u^{\epsilon_1+\epsilon_2+...+\epsilon_n}$ for all $n = 1, 2, ..., N$. If the buyer exercises his option at time n, then his payoff is $\beta^n Y_n$. From the general pricing theory of American type options, the option fair price at time 0 is

$$V_N(1) = max\{E(\alpha^\tau \beta^\tau Y_\tau)|0 \leq \tau \leq N\},$$

where τ is a stopping time and E is the expectation with respect to the martingale measure, i.e., $\epsilon_1, \epsilon_2, ..., \epsilon_N$ are i.i.d. random variables such that

$$E(\alpha u^{\epsilon_i}) = p\alpha u + (1-p)\alpha/u = 1.$$

Since $X_n = Y_n/S_n$, it is easy to see that X_n takes values in the set $\{1, u, u^2, ..., u^n\}$ and

$$V_N(1) = max\{E^*(\beta^\tau X_\tau|X_0 = 1)|0 \leq \tau \leq N\},$$

where E^* is the expectation with respect to the new probability measure P^*. With respect to this new probability measure P^*, $\epsilon_1, \epsilon_2, ..., \epsilon_N$ are i.i.d. random variables such that $P^*(\epsilon_i = 1) = p^* = \alpha up$ and $P^*(\epsilon_i = -1) = 1 - p* = \alpha(1-p)/u$. It is easy to check that $p^* = \alpha up = (u-\alpha)/(u-1/u) > 1/2$ since $u > 1 + r = 1/\alpha$.

Notice that $X_{n+1} = max\{X_n/u^{\epsilon_{n+1}}, 1\}$ for all $n = 0, 1, 2, ..., N-1$. So we can compute $V_N(1)$ by the following backward induction procedure: For $n = 0, 1, 2, ..., N$ and $k = 0, 1, 2, ..., n$, let

$$G_N(n,k) = max\{E^*(\beta^n X_n|X_n = u^k), E^*(\beta^\tau X_\tau|n \leq \tau \leq N, X_n = u^k)\}.$$

Since $G_N(N,k) = \beta^N u^k$ for all $k = 0, 1, 2, ..., N$. For all $n = 0, 1, 2, ..., N-1$ and $k = 0, 1, 2, ..., n$,

$$G_N(n,k) = max\{\beta^n u^k, p^* G_N(n+1, max(0, k-1)) + (1-p^*)G_N(n+1, k+1)\}.$$

It is clear that $V_N(1) = G_N(0,0)$ since we assume that $S_0 = 1$. This procedure requires $O(N^2)$ computations to compute the option price $G_N(0,0)$ at time 0 (the node $(0,0,0)$). However, this procedure can not produce the option prices at other accessible nodes. Theorem 5 below gives an elegant algorithm which computes the optimal execution boundary and the option price at time 0 in order N computations. Theorem 6 below gives another algorithm which computes the option prices at all possible accessible nodes time in order N^2 computations.

To determine the optimal execution time for the buyer who bought the option at time 0, Kramkov and Shiryaev gave the following algorithm. For each $n = 1, 2, ...,$ each x in the set $H = \{1, u, u^2, ..., \}$, and each function f defined on H let $Q_\beta f(x) = max\{f(x), \beta[p^* f(max(1, x/u)) + (1 - p^*)f(xu)]\}$, and let $Q_\beta^n f(x) = Q_\beta[Q_\beta^{n-1} f(x)]$. Then $V_n(x) = Q_\beta^n g(x)$, where $g(x) = x$. The optimal execution time $\tau_{N,0}$ for the buyer who bought the option at time 0 is given by $\tau_{N,0} = min\{n|0 \le n \le N, V_{N-n}(X_n) = X_n\}$. This procedure is complicated and slow. The procedure given in Theorem 5 below is much easier and faster.

For each x in H, let $V(x) = lim_{n\to\infty} V_n(x)$. Kramkov and Shiryaev showed that $V(x) = sup\{E_x^*(\beta^\tau X_\tau)|0 \le \tau < \infty\}$ and that there exists a $x_0 = u^{k_0}$ such that $V(x) = x$ for all x in H and $x \ge x_0$. Since $Q_\beta f(x) \ge f(x)$, $V_{n+1}(x) = Q_\beta[V_n(x)] \ge V_n(x) \ge x$ for all x in H. Since $V(x) = lim_{n\to\infty} V_n(x)$, $V_n(x) = x$ for all $x \ge x_0$ and all $n = 1, 2, ...$ Hence $E(n, -k_0, 0) = \beta^n$ for all $n \ge k_0$. Based on this observation and Theorem 2, we have the following theorem.

Theorem 4. There exists a positive integer k_0 which depends only on β, r, and u such that for any positive integer N, $t_n \le k_0$ for all $n = 1, 2, 3, ..., N$.

Notice that $G_N(n, k) = \beta^n u^k$ if and only if $k \ge t_n$. Based on Theorem 2, we can use the following backward induction procedure to compute $t_{N-1}, t_{N-2}, ..., t_1$ and $V_N(1)$. First let $t_{N-1} = 1$, $G_N(N-1, 0) = p^*\beta^N + (1-p^*)\beta^N u$, and $G_N(N-1, 1) = \beta^{N-1}u$. Next let $t_{N-2} = t_{N-1}$ if $\beta^{N-2}u \ge p^* G_N(N-1, 0) + (1-p^*)\beta^{N-1}u^2$, $t_{N-2} = t_{N-1} + 1$ otherwise. Let $G_N(N-2, k) = p^* G_N(N-1, max(0, k-1)) + (1-p^*)G_N(N-1, k+1)$ for $k = 0, t_{N-2} - 1$ and $G_N(N-2, t_{N-2}) = \beta^{N-2}u^{t_{N-2}}$. Now suppose that we have computed $1 = t_{N-1} \le t_{N-2} \le ... \le t_{n+1}$ for some positive integer $n_N \le n \le N-2$ and have computed all $G_N(m, k)$ for all $m = n+1, n+2, ..., N-1$ and $k = 0, 1, 2, ..., t_m$. Let $t_n = t_{n+1}$ if $\beta^n u^{t_{n+1}} \ge p^* G_N(n+1, t_{n+1} - 1) + (1-p^*)\beta^{n+1}u^{t_{n+1}+1}$, $t_n = t_{n+1} + 1$ otherwise. Let $G_N(n, k) = p^* G_N(n+1, max(0, k-1)) + (1-p^*)G_N(n+1, k+1)$ for all $k = 0, 1, 2, ..., t_n - 1$ and $G_N(n, t_n) = \beta^n u^{t_n}$. By this procedure, we will construct the the sequence $\{t_{n_N}, t_{n_N+1}, ..., t_{N-1}\}$. For $n = 0, 1, 2, ..., n_N - 1$, let $G_N(n, k) = p^* G_N(n+1, max(0, k-1)) + (1-p^*)G_N(n+1, k+1)$ for all $k = 0, 1, 2, ..., n$. The option price $V_N(1)$ at time 0 is equal to $G_N(0, 0)$, the optimal execution boundary for the buyer will be $\{u^{t_0}, u^{t_1}, u^{t_2}, ..., u^{t_{N-1}}, 1\}$. Notice that $n < t_n$ for $n = 0, 1, 2, ..., n_N - 1$. Since t_n is bounded for all $n = 1, 2, ..., N-1$, and on each step, we need only at most $t_n + 2$ computations, the total computations will be O(N). Therefore, we have the following theorem.

Theorem 5. The procedure described above requires in order N computations to compute the optimal execution boundary $\{u^{t_0}, u^{t_1}, u^{t_2}, ..., u^{t_{N-1}}, u^{t_N}\}$ and the option price $V_N(1)$ at time 0.

Based on Theorems 4 and 5, we can use the following algorithm to compute $E(n,j,k)$ for all accessible nodes (n,j,k). First let $E(N,j,k) = \beta^N u^k$ for all $k = 0,1,2,...,N$ and $j = 2k-N, 2k-N+2, ..., \leq k$. Then for $n = 0,1,2,...,N-1$ and $k = 0,1,2,...,n$, let $E(n,j,k) = \beta^n u^k$ for all $j = 2k-n, 2k-n+2, ..., \leq k-t_n$ and $E(n,j,k) = \alpha[p(n+1, j+1, max(j+1,k)) + qE(n+1, j-1, k)]$ for all $k-t_n < j \leq k$ and accessible (n,j,k). For $n = 0,1,2,...,N-1$, there are only $n(t_n+2)/2$ computations. Since $t_0, t_1, t_2, ..., t_{N-1}$ are bounded, we need only $O(N^2)$ computations to compute $E(n,j,k)$ for all $O(N^3)$ accessible nodes. Therefore, we have the following theorem.

Theorem 6. The procedure described above requires only $O(N^2)$ computations to compute all option values $E(n,j,k)$ at all $O(N^3)$ accessible nodes.

Theorems 3 and 5 show that for fixed β, r, and u, the optimal execution time depends only on n and X_n and the optimal execution boundary is of a monotonic form and bounded. These facts are exploited in our algorithms for computing the option prices and the optimal execution boundary.

Notice that for each $n = 0,1,2,...,N$, there are $n+1$ possible values for X_n. So there are $(N+1)(N+2)/2$ entries for the whole table $V_{N-n}(X_n), X_n = 1, u, u^2, ..., u^n$, and $n = 0,1,2,...,N$. We either use this table to find the optimal execution time for each price process $\{S_0, S_1, S_2, ..., S_N\}$. Or we compute $V_{N-n}(X_n)$ each time when we get a new X_n, suppose we did not exercise the option yet, then check whether $V_{N-n}(X_n) = X_n$ or not. It requires $O((N-n)^2/2)$ computations. In this respect, our approach is faster and simpler since we need only keep $N+1$ entries of u^{t_n}. For each price process $\{S_0, S_1, S_2, ..., S_N\}$, we can update X_n easily, then decide we should exercise the option or not right away.

In fact, noticing that if $x = u^k$ for some non-negative integer k, $V_{N-n}(x) = max\{E^*(\beta^\tau X_\tau | X_0 = x) | 0 \leq \tau \leq N-n\} = G_{N-n}(0,k)$. By Theorem 5, we can have a procedure which requires only $O(N-n)$ computations to compute $V_{N-n}(x)$. It will be more efficient than the procedure $V_{N-n}(x) = Q_\beta^{N-n} g(x)$ which requires $O((N-n)^2/2)$ computations.

Suppose that we have computed the set $\{t_1, t_2, ..., t_N\}$ by the backward induction procedure described in Theorem 5 for the option with expiration time N and let $\{t'_1, t'_2, ..., t'_{N+M}\}$ be the counter part for the option with expiration time $N+M$, where M is a positive integer. Do we have to compute the entire set $\{t'_1, t'_2, ..., t'_{N+M}\}$? The following theorem reveals that there is a relationship between the set $\{t_1, t_2, ..., t_N\}$ and the set $\{t'_1, t'_2, ..., t'_{N+M}\}$.

Theorem 7. Let $\{t_1, t_2, ..., t_N\}$ and $\{t'_1, t'_2, ..., t'_{N+M}\}$ be as defined above. Then

$t'_{N+M-j} = t_{N-j}$ for all $j = 0, 1, 2, ..., N - n_N$.

Proof: We will prove Theorem 7 when $M = 1$. The general case can be proved by mathematical induction.

Notice that $G_{N+1}(N+1, k) = \beta^{N+1}u^k$ and $G_N(N, k) = \beta^N u^k$ for all $k = 0, 1, 2, ..., N$. Also notice that

$$G_N(n, k) = max\{\beta^n u^k, p^* G_N(n+1, max(0, k-1)) + (1-p^*)G_N(n+1, k+1)\}$$

and

$$G_{N+1}(n+1, k)$$
$$= max\{\beta^{n+1}u^k, p^* G_{N+1}(n+2, max(0, k-1)) + (1-p^*)G_{N+1}(n+2, k+1)\}$$

for all $k = 0, 1, 2, ..., N$ and $n = 0, 1, 2, ..., N-1$. By mathematical induction, it is easy to see that $G_{N+1}(n+1, k) = \beta G_N(n, k)$ for all $k = 0, 1, 2, ..., n$ and $n = n_N, n_N + 1, ..., N$. $G_N(n, k) = \beta^n u^k$ if and only if $G_{N+1}(n+1, k) = \beta^{n+1}u^k$ for $n = n_N, n_N+1, ...$ It implies that $t_N = t'_{N+M}, t_{N-1} = t'_{N+M-1}, ..., t_{n_N} = t'_{n_N+M}$.

Based on Theorem 7, we can use the optimal execution boundary of an option and some additional computations to get the optimal execution boundary for a new option with a longer expiration time. Conversely we can use the tail segment of the optimal execution boundary of an option for the optimal execution boundary of a new option with a shorter expiration time. This is another nice property of the approach presented in this paper. Theorem 7 also confirms that if the time horizon is infinite, then the optimal execution boundary is a horizontal line and the optimal execution time for the buyer is the first n such that $X_n = u^{t_1}$ since $t_1 = t_2 = ... = b$ for some positive constant b if the time horizon is infinite. The results presented in this paper are consistent with the results in [6] by Shepp and Shiryaev.

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