INFERING MODEL PARAMETERS IN MARKETS WITH COLLARS

Robert W. Chen
Burton J. Rosenberg
University of Miami

Yi-Tsung Lee
National Chung Cheng University

1 Introduction

Security prices are set by a continuous auction. It is assumed that under these circumstances all relavant information will quickly incorporate itself to the stock price. As a result of this activity, the logarithm of stock prices move randomly according to a Weiner-Levy process, perhaps with an added component of drift depending on the season, investor psychology, or other poorly understood factors. This model of stock prices has been very important both theoretically and practically to the development of option pricing.

However, many markets have collars, under which the rules of the auction will not let prices move too rapidly. We consider the case where the previous day's closing price is set as the center of a percentage band outside of which the stock will not trade for that day. The question arises as to the proper pricing of options in such markets.

We propose a simply model in which there as an "ideal" stock price, — the market clearing price which may be outside of the legally allowed price range. The observed market trace is this ideal price subject to a collering algorithm which keeps daily price movements within band. We also assume that the market price will always return to the ideal price, so that investors will always have the opportunity to trade at the ideal price, perhaps after having to waiting a few days. Conditions on collar size can insure that this is so

Our proposition, then is that options in markets with collars should be priced according to the standard Black-Scholes applied to the ideal price. of the ideal price from the observered price. In this paper we present three methods by which this can be done, give proofs of strong consistency and compare their accuracy. These methods are,

1. a method based on the measure of waiting times;
2. a method based on the estimate of the likelihood of an at-market day being followed by an at-limit day (definitions follow);
3. the generalized moment method introduced by Chiang and Wei [2].

The method of generalized moments is due to Chiang and Wei [2], however their proof of strong consistency lacked rigor.

In addition, we compare the results of these methods to the volatility of the actual stock price and perform simulations on which of the two values are more appropriate for option pricing.

2 Definitions

Our ideal stock trace is a sequence of closing prices $S_0, S_1, \ldots$. The interday increments, expressed logarithmically, are independent, identically distributed random variables with normal distribution with common mean μ and variance σ^2,

$$Z_i = \ln(S_i/S_{i-1}) \sim \mathcal{N}(\mu, \sigma^2).$$

This is the standard model of stock prices [1].

Collars are applied to the ideal trace S_i to create an observable sequence S_i^* of prices. Given lower and upper collars,

$$\kappa_l < 1 < \kappa_u,$$

we define S_i^*, $i = 0, 1, 2, \ldots$ according to,

$$S_i^* = \begin{cases} S_0 & \text{if } i = 0, \\ \kappa_l\, S_{i-1}^* & \text{if } S_i < \kappa_l\, S_{i-1}^*, \\ \kappa_u\, S_{i-1}^* & \text{if } S_i > \kappa_u\, S_{i-1}^*, \\ S_i & \text{otherwise.} \end{cases}$$

We classify each observation S_i^* as either *at-market* or *at-limit*. Since we cannot always observe S_i, an observation S_i^* will be considered at-market only if it is strictly within that observation's collar. Else we must conclude that the observation is at-limit. Formally, the set of market observations is,

$$T = \{i > 0 \mid (\kappa_l\, S_{i-1}^*) < S_i^* < (\kappa_u\, S_{i-1}^*)\} \cup \{0\}$$

and the set of limit observations is the complement,

$$L = T^c = \{i > 0 \mid S_i^* = (\kappa_l\, S_{i-1}^*) \text{ or } S_i^* = (\kappa_u\, S_{i-1}^*)\}.$$

We enumerate the indices of T and L in ascending order as $T_0, T_1, \ldots$ and $L_0, L_1, \ldots$. Note that by construction $T_0 = 0$, and that L might be empty.

3 The waiting time method

The manner by which we estimate the mean μ of S_i is the same in all the three methods discussed. If we consider any two consecutive at-market observations, we have a random variable,

$$X_i = \ln\left(S_{T_i}/S_{T_{i-1}}\right) = \sum_{j=T_{i-1}+1}^{T_i} Z_j.$$

Since Z_i and Z_j are independent for $i \neq j$, the X_i are independent identically distributed random variables. By the Wald Equation [3],

$$E(X_1) = E(Z_1)E(T_1) = \mu E(T_1).$$

Using the sample expectations,

$$\begin{aligned} E(X_1) &\approx (1/n)\sum_{i=1}^{n} X_i \\ E(T_1) &\approx (1/n)\sum_{i=1}^{n}(T_i - T_{i-1}) = T_n/n \end{aligned}$$

we have,

$$\mu = E(X_1)/E(T_1) = (\ln S_{T_n} - \ln S_0)/T_n$$

where $n = |T|$ is the number of days observed at market in our sample.

To estimate the variance, we look at the waiting time for the next at-limit observation begining from an at-market observation. The probabilty of hitting the collar is given by the weight of the tails of the normal distribution function integrated outside the region $\ln \kappa_l < x < \ln \kappa_u$. The waiting time is the reciprocal of this probability. This gives us the following implicit relation between $E(W)$ and σ^2,

$$1 - \frac{1}{E(W)} = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\ln \kappa_l}^{\ln \kappa_u} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx.$$

The expectation of W can be sampled by looking at the observations S_i^*. Define T' to be the first at-market day in a sequence of at market days, and L' to be the first at-limit day in a sequence of at-limit days,

$$\begin{aligned} T' &= \{i \in T \,|\, i > 1,\, i-1 \notin T\} \cup \{0\} \\ L' &= \{i \in L \,|\, i-1 \in T\} \end{aligned}$$

Enumerate these indices as $T'_1, T'_2, \ldots, T'_m$ and $L'_1, L'_2, \ldots, L'_m$. Pairing them up we calculate the i-th waiting time,

$$W_i = L'_i - T'_i.$$

The expected waiting time is estimated by the mean of these observed waiting times,

$$E(W) \sim (1/m)\sum_{i=1}^{m} W_i.$$

Using numerical methods, we solve for σ^2 using this sample value of $E(W)$ and the implicit integral relation between $E(W)$ and σ^2.

4 The transition counting method

As refered to in the previous section, we can view the outcome of what follows an at-market day as the flip of a biased coin. The bias on this coin relates to the area under the gaussian distribution which might put the next market step into the limit region — and this is related monotonically to the variance. By observing the ratio of at-market days followed by at-market days versus by at-limit days, this bias can be estimated.

Define R to be the set of indices such that two consecutive days are at market and S to be the set of indices such that the first day is at market and the next is at limit,

$$\begin{aligned} T_{MM} &= \{i \,|\, (i \in T) \wedge (i+1 \in T)\}, \\ T_{ML} &= \{i \,|\, (i \in T) \wedge (i+1 \in L)\}. \end{aligned}$$

We are flipping a coin which comes up "limit" with probability $|T_{ML}|/(|T_{ML}| + |T_{MM}|)$. Therefore,

$$E(W) = 1 + (|T_{MM}|/|T_{ML}|).$$

This formula agrees with the formulat for $E(W)$ of the waiting time method if the sequence of observations ends on an at-limit day. Else the estimate of $E(W)$ will be slightly lower, as the sequence of ending observations will bias downward the estimation of the probability of hitting the limit.

Using numerical methods, we solve for σ^2 using this sample value of $E(W)$ and the implicit integral relation between $E(W)$ and σ^2, as in the waiting time method.

5 The generalized moment method

Both expection and variance are linear when sums are taken of independent random variables. Therefore there is hope that we could distribute over the at-limit days which intervene between two at-market days the volatility observed between these two market days.

should we call S_{T_i} to $S_{T_{i+1}}$ one observation or $T_{i+1} - T_i$ observations? The GMM does the later. I am not sure if this is correct.

Intuitively, if S_i and S_j are two at-market observations, then the common variance of $Z_{i+1}, Z_{i+2}, \ldots, Z_j$ is summed to achieve the variance of $S_j - S_j$. If we measure the variance of $S_j - S_i$ and divide by $j - i$, then we have the variance of each $Z_\cdot$. To form the sample variance over the entire suite $Z_0, \ldots, Z_n$ we shall again multiply by $j - i$. This motivates the formula:

$$\sigma^2 = (1/T_k) \sum_{i=1}^{k} (S_{T_k} - S_{T_{k-1}} - \mu(T_k - T_{k-1}))^2$$

where the T_i are the indices of the at-market observations in ascending order.

The difficulty for proving this formula were not well handled in the original paper.

Appendix: Simulation in Mathematica

```
(*
   Burton Rosenberg
   6:43 PM 10/5/00

   version 4
*)

(* include packages *)
<< Statistics‘ContinuousDistributions‘
<< Statistics‘DescriptiveStatistics‘

(* trace generation routines *)

weinerWalk[ n_, m_, sd_ ] :=
    Module [
        { s = 0, sl = {0} },
        Do [
            s = s + Random[NormalDistribution[m, sd]] ;
            sl = Append[sl, s]
            , {n}
        ] ;
        sl
    ]

collarX[x_, s_, l_, u_ ] :=
    If[ ( x > (s+u) ),
        (s+u) ,
        If[ (x < (s-l) ),  (s-l), x ]
    ]

applyCollar[ walk_, l_, u_ ] :=
    Module [
        { walkL = {}, x = walk[[1]] },
        Do [
            x = collarX[ walk[[i]], x, l, u ] ;
            walkL = Append[ walkL, x ]
            , {i, 1, Length[walk]}
        ] ;
```

```
        walkL
    ]

classifyDays[ a_, b_ ] :=
    Module [
        { c = {} },
        Do [
            If [ ( a[[i]]==b[[i]] ),
                c = Append[ c, 0 ],
                c = Append[ c, 1 ]
            ]
            , {i, 1, Length[a]}
        ] ;
        c
    ]

generateTrace[ n_, mean_, sd_, lower_, upper_] :=
    Module [
        { rw, cw, cd },
        rw =  weinerWalk[ n, mean, sd ] ;
        cw = applyCollar[ rw, lower, upper ] ;
        cd = classifyDay[ rw, cw ] ;
        (* return { ideal walk, observed walk, days classified } *)
        (* codes for days classified: 0==at-market, 1==at-limit *)
        { rw, cw, cd }
    ]


(* end trace generation routines *)

(* helper functions to estimation routines *)

countTransitions[ daysClassified_ ] :=
    Module [
        { result = { 0, 0, 0, 0 },  index },
        Do [
          If[ ( daysClassified[[i]]==0 ), index=1, index=3 ] ;
          If[ ( daysClassified[[i+1]]==0 ),
                      index=index, index=index+1 ] ;
          result[[index]]++
          , {i, 1, Length[daysClassified]-1}
        ] ;
        (* { marketMarket, marketLimit,
             limitLimit, limitMarket } *)
```

```
        result
    ]

waitingTimes[ daysClassified_ ] :=
    Module [
       { waitT = {}, classOfPreviousDay, currentWait=0 },
       (* assert: daysClassified[[1]]==0 *)
       Do [
           If[ (daysClassified[[i]]==0),
               currentWait++ ;
               , (* else *)
               If [ (classOfPreviousDay==0),
                   waitT = Append[ waitT, currentWait ] ;
                   currentWait = 0
               ]
           ] ;
           classOfPreviousDay = daysClassified[[i]]
           , {i, 1, Length[daysClassified]}
       ] ;
       waitT
    ]

lastMarketDay[ daysClassified_ ] :=
    Module [ { lmd },
        (* assert: daysClassified[[1]]==0 *)
        Do [
            If [ (daysClassified[[i]]==0), lmd = i ]
            , {i, 1, Length[daysClassified] }
        ] ;
        lmd
    ]


f[ mu_, sigma_, ll_, ul_ ] :=
    1 - Integrate[
           ( 1 / (sigma Sqrt[ 2 Pi ]) )
                     Exp[ - (t-mu)^2 / ( 2 sigma^2 ) ],
           { t, ll, ul }
        ]

(* estimation routines *)

estimateMean[ aTrace_ ] :=
    Module[ { lmd = lastMarketDay[ aTrace[[3]] ] },
```

```
        (*note: arrays count from 1, differs from text *)
        (aTrace[[1]])[[lmd]]/(lmd-1)
    ]

waitingTimeMethod[ aTrace_, lowerCollar_, upperCollar_ ] :=
    Module[ { wt, p },
       wt = waitingTimes[ aTrace[[3]] ] ;
       (* fix case of no limit days *)
       If [ (Length[wt]==0),
           wt = { Length[aTrace[[1]]] } ] ;
       p = 1 / Mean[ wt ] ;
       FindRoot[ f[estimateMean[ aTrace ], x,
           -lowerCollar, upperCollar]==p, {x,1},
           MaxIterations->100, AccuracyGoal->15 ]//N
    ]

transitionCountingMethod[ aTrace_, lowerCollar_, upperCollar_ ] :=
    Module[ { cx, p },
        cx = countTransitions[ aTrace[[3]] ] ;
        p = cx[[2]]/(cx[[1]]+cx[[2]]) ;
        FindRoot[ f[estimateMean[ aTrace ], x,
            -lowerCollar, upperCollar]==p, {x,1},
            MaxIterations->100, AccuracyGoal->15 ]//N
    ]

generalizedMomentsMethod[ aTrace_ ] :=
    Module [
        {   data = aTrace[[1]],
            dayType = aTrace[[3]],
            mu = estimateMean[ aTrace ],
            listLength = Length[ aTrace[[1]] ],
            sum, i, j
        } ,
        (* loop invariant, sum contains all data up to day i;
           day i is a market day *)
        sum = 0 ;
        i = 1 ;
        While [ (True),
           j = i+1 ;
           While [ (j<=listLength && dayType[[j]]==1), j++ ] ;
           If [ (j>listLength), Break[] ] ;
           sum += ((data[[j]]-data[[i]])-(j-i)*mu)^2 ;
           i = j ;
        ] ;
```

```
        Sqrt[sum/listLength]
    ]

observedMoments[ aTrace_ ] :=
    Module [
        {  data = aTrace[[2]],
           mu, sum
        },
        sum = 0 ;
        mu = data[[Length[data]]]/(Length[data]-1) ;
        Do [
            sum += ( data[[i+1]]-data[[i]]-mu )^2 ;
            ,{ i, 1, Length[data]-1 }
        ] ;
        { mu, Sqrt[sum/(Length[data]-2)] }
    ]


(* sample runs *)

runAll[ n_, m_, sd_, lc_, uc_ ] :=
    Module[ { theTrace },
        theTrace = generateTrace[ n, m, sd, lc, uc ] ;
        { estimateMean[ theTrace],
          waitingTimeMethod[ theTrace, lc, uc ],
          transitionCountingMethod[ theTrace, lc, uc ],
          generalizedMomentsMethod[ theTrace ],
          observedMoments[ theTrace ]
        }
    ]
```

References

[1] Robert Cast, André Lapied *Fondements Microéconomique de la Théorie des Marchés Financiers,* Economica, Paris 1992.

[2] R. Chiang, K. Wei, *GMM and MLE approaches for estimation of volatility and regression models when daily prices are subject to price limits*, manuscript.

[3] William Feller, *An Introduction to Probability Theory and its Applications*, Volume 2, John Wiley and Sons, 1966.

[4] John C. Hall, *Options, Futures, and Other Derivatives,* Prentice-Hall, 1997.