Inferring model parameters in markets with collars

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Abstract

Security prices are set by a continuous auction, the rules of which are set by the exchange or by the government. For many exchanges, there is a general free-flow of price information resulting in stock prices which can be modelled by a random walk following a Weiner-Levy process. However, many markets have collars, under which the rules of the auction will not let prices move too rapidly. In this paper we present methods for estimating the volatility of the underlying price data when the true price information is obscured by such collars. Numerical simulations are presented which demonstrate and contrast the methods.

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1 Introduction

Security prices are set by a continuous auction. The rules of the auction are set by the exchange or by the government. For many exchanges, there is a general free-flow of price information resulting in stock prices which can be modelled by a random walk following a Weiner-Levy process, perhaps with an added component of drift depending on the season, investor psychology, or other poorly understood factors. This model of stock prices has been very important both theoretically and practically to the development of option pricing.

However, many markets have collars, under which the rules of the auction will not let prices move too rapidly. We consider the case where the previous day's closing price is set as the center of a percentage band outside of which the stock will not trade for that day. Since this affects the volatility of stock prices, and options are priced as a function of volatility, the question arises as to the proper pricing of options in markets with collars.

We propose a simply model in which there is an ideal stock price and an exchange price. The ideal stock price is the market clearing price which may be outside of a legally allowed price range. The exchange price is the price quoted by the exchange and is subject to the exchange or government restrictions on price ranges. We assume that the ideal or market-clearing price is not affected by the exchange price, that the exchange price is derived by applying a deterministic algorithm to the ideal price. We also assume that the exchange price will always return to the ideal price, so that investors will always have the opportunity to trade at the ideal price. Conditions on collar size can insure that this is so.

In this paper we present three methods by which the volatility of the ideal price can be estimated by looking only at the exchange price:

1. a method based on the measure of waiting times;

2. a method based on the estimate of the likelihood of an at-market day being followed by an at-limit day (definitions follow);

3. the renewal time method introduced by Chiang and Wei.

The renewal time method is due to Chiang and Wei. In this paper we provide a rigorous proof of strong consistency for their method.

In addition, we compare the results of these methods to the volatility of the actual stock price and perform simulations on which of the two values are more appropriate for option pricing.

2 Methods

2.1 Definitions

Our ideal stock trace is a sequence of closing prices $S_0, S_1, \ldots$. The inter-day increments are independent, identically distributed random variables with log-normal distribution of common mean μ and variance σ^2,

$$Z_i = \ln(S_i/S_{i-1}) \sim \mathcal{N}(\mu, \sigma^2).$$

If r is the risk-free rate, then $\mu = r - \sigma^2/2$.

Collars are applied to the ideal price trace S_i to create an observable sequence S_i^* of exchange prices. Given lower and upper collars,

$$\kappa_l < 1 < \kappa_u,$$

we define S_i^*, $i = 0, 1, 2, \ldots$ according to,

$$S_i^* = \begin{cases} S_0 & \text{if } i = 0, \\ \kappa_l \, S_{i-1}^* & \text{if } S_i < \kappa_l \, S_{i-1}^*, \\ \kappa_u \, S_{i-1}^* & \text{if } S_i > \kappa_u \, S_{i-1}^*, \\ S_i & \text{otherwise.} \end{cases}$$

We classify each observation S_i^* as either *at-market* or *at-limit*. Since we cannot always observe S_i, an observation S_i^* will be considered at-market only if it is strictly within that observation's collar. Else we must conclude that the observation is at-limit. Formally, the set of market observations is,

$$T = \{i > 0 \,|\, (\kappa_l \, S_{i-1}^*) < S_i^* < (\kappa_u \, S_{i-1}^*)\} \cup \{0\}$$

and the set of limit observations is the complement,

$$L = T^c = \{i > 0 \,|\, S_i^* = (\kappa_l \, S_{i-1}^*) \text{ or } S_i^* = (\kappa_u \, S_{i-1}^*)\}.$$

We enumerate the indices of T and L in ascending order as $T_0, T_1, \ldots$ and $L_0, L_1, \ldots$. Note that by construction $T_0 = 0$, and that L might be empty.

An example trace of 90 day data is given in figure 1. The ideal price, exchange price and their difference are plot, with the difference trace shifted and scaled for display. The simulated stock has zero mean, $\sigma = 47\%$, and the collars were set to $(5/4)\,\sigma$.

2.2 Estimation of the mean

We estimate the mean μ of S_i and prove some additional lemmas.

If we consider any two consecutive at-market observations, we have a random variable,

$$X_i = \ln\left(S_{T_i}/S_{T_{i-1}}\right) = \sum_{j=T_{i-1}+1}^{T_i} Z_j.$$

Since Z_i and Z_j are independent for $i \neq j$, the X_i are independent identically distributed random variables. This is also true that the random variable $T'_i = T_i - T_{i-1}$ are independent and identically distributed.

Define $B_k(a)$ to be the event that the ideal price is above the upper limit of the exchange price for at least k consecutive days given that at day zero the ideal price is a above the zero-th day upper limit,

$$B_k(a) \quad = \quad \{\, (a + Z_1 + \ldots + Z_j) > j \ln \kappa_u, \text{ for } j = 1, \ldots, k \,\},$$

Lemma 1 *Let* $\lambda = (\ln \kappa_u - \mu)/\sigma$. *If* $\lambda > 0$ *then for large enough* k *and* $a \geq 0$*,*

$$Prob\,(B_k(a)) < \frac{e^{-k\lambda^2/4}}{\lambda\sqrt{\pi k}}.$$

PROOF: To remain above the exchange limit for k steps requires that the current ideal price exceeds by $k \ln \kappa_u$ the exchange price k days earlier. Since,

$$Y = a + Z_1 + \ldots + Z_k \sim \mathcal{N}(a + k\mu, k\sigma^2),$$

then,

$$\begin{aligned}\text{Prob}\,(\,Y > k\ln\kappa_u\,) &= \frac{1}{\sqrt{2\pi k\sigma^2}}\int_{k\ln\kappa_u}^{\infty} e^{-(x-k\mu-a)^2/(2k\sigma^2)}\,dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{(\lambda\sqrt{k})-a/(\sigma\sqrt{k})}^{\infty} e^{-x^2/2}\,dx.\end{aligned}$$

Because $\lambda > 0$, we can select a K such that for all $k \geq K$,

$$(\lambda\sqrt{k}) - a/(\sigma\sqrt{k}) > \lambda\sqrt{k/2} > 0$$

and we can use the approximation for the tail of the normal distribution,

$$\frac{1}{\sqrt{2\pi}}\int_x^{\infty} e^{-y^2/2}\,dy < \frac{1}{x\sqrt{2\pi}}\,e^{-x^2/2}$$

for $x > 0$, see Feller, Vol. 1, Lemma VII.1.7. Since the bound is increasing with decreasing x, for simplicity of notation we set $x = \lambda\sqrt{k/2}$ to complete the proof. $\triangle$

Lemma 2 *If* $\ln\kappa_l < \mu < \ln\kappa_u$ *then* $E(T_1)$ *and* $E(T_1^2)$ *are bound.*

Proof: A sequence of days which stay outside of the market will consist of segments of days in one collar alternating with segments of days in the other collar.

Consider now the random variable B^u which gives the number of days we can stay within the upper collar. Note that $B_k^u = \bigcap_{j=1}^k B_j$. Using the previous lemma,

$$\begin{aligned}E(B^u) &< \sum_{k=0}^{\infty}\text{Prob}\,(B_k) \\ &< C_N + \sum_{k=N}^{\infty}\frac{e^{-k\lambda^2/4}}{\lambda\sqrt{\pi k}} < \infty,\end{aligned}$$

where $\lambda = (\ln\kappa_u - \mu)/\sigma$. An similar bound exists for expectation time in the lower collar $E(B^l)$.

Now consider the random variable C which gives the number of passages from upper to lower collars before the next consecutive day at-market. The probability to pass from one collar to the other must be at least,

$$p_c = \text{Prob}\,(\,Z_j \geq (\ln\kappa_u - \ln\kappa_l)\,).$$

Considering the various cases by which a random walk might stay outside of either collar, we arrive at, essentially, a sum over i changes from upper to lower of walks of approximate length weighted by the probability of i changes of collars,

$$\begin{aligned} E(T_1) &\approx \text{Prob}\,(\,\ln\kappa_l < Z_1 < \ln\kappa_u\,) + \sum_{i=1}^{\infty}(p_c^{i-1})(i/2)(E(B^u) + E(B^l) + i) \\ &< K_c + K\sum_{i=1}^{\infty} i^2 p_c^i < \infty. \end{aligned}$$

The proof for $E(T_1^2)$ is similar. △

Lemma 3 $E(X_{T_1}) = \mu E(T_1)$.

PROOF: It is now more convenient to work with increments $\mu + \sigma Z_i$ with respect to normalized random variables Z_i.

$$\begin{aligned} E(X_{T_1}) &= E\left(\sum_{j=1}^{T_1}\mu + \sigma Z_j\right) \\ &= \sum_{k\geq 1}\int_{\{T_1=k\}}\sum_{j=1}^{k}(\mu + \sigma Z_j)\,d\mathcal{P} \\ &= \sum_{k\geq 1} k\mu P(T_1 = k) + \sigma\sum_{k\geq 1}\sum_{j=1}^{k}\int_{\{T_1=k\}} Z_j\,d\mathcal{P} \end{aligned}$$

The first term on the right-hand side is obviously $\mu E(T_1)$. Rearranging the sum of the second term on the right hand side,

$$\begin{aligned} \sigma\sum_{k\geq 1}\sum_{j=1}^{k}\int_{\{T_1=k\}} Z_j\,d\mathcal{P} &= \sigma\sum_{j\geq 1}\sum_{k\geq j}\int_{\{T_1=k\}} Z_j\,d\mathcal{P} \\ &= \sigma\sum_{j\geq 1}\int_{\{T_1\geq j\}} Z_j\,d\mathcal{P} \\ &= \sigma\sum_{j\geq 1}\left(E(Z_j) - \int_{\{T_1<j\}} Z_j\,d\mathcal{P}\right) \end{aligned}$$

Since Z_j and $\{T_1 < j\}$ are independent,

$$E(Z_j) - \int_{\{T_1<j\}} Z_j\,d\mathcal{P} = E(Z_j)(1 - P(T_1 < j)) = 0$$

because $E(Z_j) = 0$. We conclude that the second term is zero. △

Since $E(X_{T_1})$ and $E(T_1)$ are finite, the Strong Law of Large Numbers gives us strong consistency for the sample expectations of of X_{T_1} and T_1,

$$\widehat{E(X_{T_1})} = (1/n)\sum_{i=1}^{n} X_i$$

$$\widehat{E(T_1)} = (1/n)\sum_{i=1}^{n}(T_i - T_{i-1}) = T_n/n$$

and a strongly consistent approximation $\hat{\mu}$ to μ,

$$\hat{\mu} = E(X_{T_1})/E(T_1) = (\ln S_{T_n} - \ln S_0)/T_n$$

where $n = |T|$ is the number of days observed at-market in our sample.

Theorem 1 *Under the given assumptions $\hat{\mu}$ is strongly consistent.*

2.3 Estimation of σ: the waiting time method

To estimate the variance, we look at the waiting time for the next at-limit observation beginning from an at-market observation. Define T' to be the first at-market day in a sequence of at market days, and L' to be the first at-limit day in a sequence of at-limit days,

$$\begin{aligned} T' &= \{i \in T \,|\, i > 1,\, i-1 \notin T\} \cup \{0\} \\ L' &= \{i \in L \,|\, i-1 \in T\} \end{aligned}$$

Enumerate these indices as $T'_1, T'_2, \ldots, T'_m$ and $L'_1, L'_2, \ldots, L'_m$. Pairing them up we calculate the i-th waiting time $W_i = L'_i - T'_i$. The W_i are independent identically distributed random variables with geometric distribution $\{pq^{k-1}\}$ where q is the probability that Z_i falls within the collar $\ln\kappa_l < Z_i < \ln\kappa_u$. Hence,

$$E(W_1) = 1/(1-q) < \infty.$$

This gives us the implicit relation between $E(W_1)$ and σ,

$$1 - \frac{1}{E(W_1)} = \frac{1}{\sqrt{2\pi}}\int_{(\ln\kappa_l - \mu)/\sigma}^{(\ln\kappa_u - \mu)/\sigma} e^{-x^2/2}\,dx.$$

Since $E(W_1) < \infty$, the Strong Law of Large Numbers allows that the sample mean $\widehat{E(W_1)}$ is a strongly consistent estimate for $E(W_1)$,

$$\widehat{E(W_1)} = (1/m) \sum_{i=1}^{m} W_i.$$

We use numerical methods to solve for $\hat{\sigma}$ given $\widehat{E(W_1)}$. Due to the continuity of this relation, that $\widehat{E(W_1)}$ is a strongly consistent estimate for $E(W_1)$ implies that $\hat{\sigma}$ is a strongly consistent estimate for σ.

Theorem 2 *Under the given assumptions $\hat{\sigma}$ is strongly consistent.*

2.4 Estimation of σ: the transition counting method

As referred to in the previous section, we can view the outcome of what follows an at-market day as the flip of a biased coin. The bias on this coin relates to the area under the gaussian distribution which might put the next market step into the limit region — and this is related monotonically to the variance. By observing the ratio of at-market days followed by at-market days versus by at-limit days, this bias can be estimated.

Define R to be the set of indices such that two consecutive days are at market and S to be the set of indices such that the first day is at market and the next is at limit,

$$\begin{aligned} T_{MM} &= \{i \,|\, (i \in T) \wedge (i+1 \in T)\}, \\ T_{ML} &= \{i \,|\, (i \in T) \wedge (i+1 \in L)\}. \end{aligned}$$

We are flipping a coin which comes up “limit” with probability $|T_{ML}|/(|T_{ML}| + |T_{MM}|)$. Therefore,

$$E(W) = 1 + (|T_{MM}|/|T_{ML}|).$$

This formula agrees with the formula for $E(W)$ of the waiting time method if the sequence of observations ends on an at-limit day. Else the estimate of $E(W)$ will be slightly lower, as the sequence of ending observations will bias downward the estimation of the probability of hitting the limit.

Because this method agrees so closely with the method of waiting times, we will not consider it further.

2.5 Estimation of σ: the renewal time method

Recall that our at-market days have indices $T_0, T_1, \ldots$. The renewal time from one at-market day to another is $T_i' = T_{i-1} - T_i$, for $i = 1, 2, \ldots$. The T_i' are i.i.d. random variables. The change in log market price during interval T_i' is,

$$X_{T_i'} = \sum_{j=T_{i-1}-1}^{T_i} (\mu + \sigma Z_j)$$

where the Z_i are i.i.d. standard gaussian normal random variables.

Lemma 4 $E\big((X_{T_1} - \mu T_1)^2\big) = \sigma^2 E(T_1)$.

PROOF: Since, $X_{T_1} - \mu T_1 = \sum \sigma Z_j$, it is sufficient to show that $E\big((\sum Z_j)^2\big) = E(T_1)$.

$$\begin{aligned} E\left[\left(\sum_{j=1}^{T_1} Z_j\right)^2\right] &= \sum_{k\geq 1} \int_{\{T_1=k\}} \left(\sum_{j=1}^{k} Z_j\right)^2 d\mathcal{P} \\ &= \sum_{k\geq 1}\sum_{i=1}^{k} \int_{\{T_1=k\}} Z_i^2 \, d\mathcal{P} + 2 \sum_{k\geq 1}\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} \int_{\{T_1=k\}} Z_i Z_j \, d\mathcal{P} \end{aligned}$$

Consider the first term on the right-hand side,

$$\begin{aligned} \sum_{k\geq 1}\sum_{i=1}^{k} \int_{\{T_1=k\}} Z_i^2 \, d\mathcal{P} &= \sum_{i\geq 1}\sum_{k\geq i} \int_{\{T_1=k\}} Z_i^2 \, d\mathcal{P} \\ &= \sum_{i\geq 1} \int_{\{T_1\geq i\}} Z_i^2 \, d\mathcal{P} \\ &= \sum_{i\geq 1} \left(E(Z_i^2) - \int_{\{T_1<i\}} Z_i^2 \, d\mathcal{P} \right) \\ &= \sum_{i\geq 1} E(Z_i^2)(1 - P(T_1 < i)) \end{aligned}$$

because $\{T_1 < i\} \in \mathcal{F}_{i-1}$ and is independent of Z_j. Noting that $E(Z_i^2) = 1$, we have that this first term is $\sum P(T_1 \geq i) = E(T_1)$.

The second term is rearranged and shown to equal zero,

$$\begin{aligned} 2\sum_{k\geq 1}\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}\int_{\{T_1=k\}} Z_iZ_j\,d\mathcal{P} &= 2\sum_{i\geq 1}\sum_{j\geq i+1}\sum_{k\geq j}\int_{\{T_1=k\}} Z_iZ_j\,d\mathcal{P} \\ &= 2\sum_{i\geq 1}\sum_{j\geq i+1}\int_{\{T_1\geq j\}} Z_iZ_j\,d\mathcal{P} \\ &= 2\sum_{i\geq 1}\sum_{j\geq i+1}\left(E(Z_iZ_j) - \int_{\{T_1<j\}} Z_iZ_j\,d\mathcal{P}\right) \end{aligned}$$

For $i \neq j$, $E(Z_iZ_j) = E(Z_i)E(Z_j) = 0$. Also, since $I_{\{T_1<j\}}Z_i$ is independent of Z_j when $i < j$, the integral is zero. △

Theorem 3 *The estimate,*

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(X_{T_i'} - \hat{\mu}T_i')^2}{\sum_{i=1}^{n} T_i'}$$

is strongly consistent.

PROOF: Rewriting the numerator,

$$\begin{aligned} \sum_{i=1}^{n}(X_{T_i'} - \hat{\mu}T_i')^2 &= \sum_{i=1}^{n}\big((X_{T_i'} - \mu T_i') + (\mu - \hat{\mu})T_i'\big)^2 \\ &= \sum_{i=1}^{n}(X_{T_i'} - \mu T_i')^2 + 2\sum_{i=1}^{n}(X_{T_i'} - \mu T_i')(\mu - \hat{\mu})T_i' \\ &\quad + \sum_{i=1}^{n}\big((\mu - \hat{\mu})T_i'\big)^2. \end{aligned}$$

By the previous lemma and the Law of Large Numbers, the first term of the sum converges to $n\sigma^2E(T_1)$. We have shown that the denominator converges to $nE(T_1)$. Therefore to prove $\hat{\sigma}^2$ is strongly consistent it is sufficient to show,

$$(\mu - \hat{\mu})^2\sum_{i=1}^{n} T_i'^2 / \sum_{i=1}^{n} T_i' \;\to\; 0$$

and

$$\sum_{i=1}^{n}(X_{T_i'} - \mu T_i')(\mu - \hat{\mu})T_i' / \sum_{i=1}^{n} T_i' \;\to\; 0$$

with probability 1.

We have shown that $E({T_1}^2)$ exists and is finite, and that $\mu - \hat{\mu} \;\rightarrow\; 0$ with probability 1, so the first of these limits is proven.

The second limit is shown using the Cauchy-Schwarz inequality,

$$\begin{aligned} 0 &\leq \left(\sum_{i=1}^{n}(X_{T_i'} - \mu T_i')(\mu - \hat{\mu})T_i' / \sum_{i=1}^{n} T_i'\right)^2 \\ &\leq \left(\sum_{i=1}^{n}(X_{T_i'} - \mu T_i') / \sum_{i=1}^{n} T_i'\right)\left(\sum_{i=1}^{n}(\mu - \hat{\mu})T_i' / \sum_{i=1}^{n} T_i'\right) \\ &\rightarrow \sigma^2 \cdot 0 = 0 \end{aligned}$$

with probability 1. △

3 Results

In graphs in Figures 1 through 5 plot results of the estimation of σ for the waiting time method, the renewal time method, and the direct calculation on the observed price stream, with this data displayed as box charts, left to right as listed.

For each trial, a random walk of 250 days was generated, with $\mu = 0$ and $\sigma = 1$. The upper and lower collars equal a common κ, where these collars are applied to the log data,

$$\ln \kappa_l = \ln \kappa_u = \kappa.$$

Various ratios κ/σ are tried. A plot of 100 trials with a bisected box indicating the median and one quartile above and below the median is given.

For lower κ/σ, the waiting time method shows greater accuracy (median is closer to 1) and less dispersion (quartiles are closer to median). By $\kappa/\sigma = 2$ the two methods perform about equally, and above that the renewal time method tends to excel. In this region, hitting a collar is rare, so we do not get enough waiting time events to accurately measure σ.

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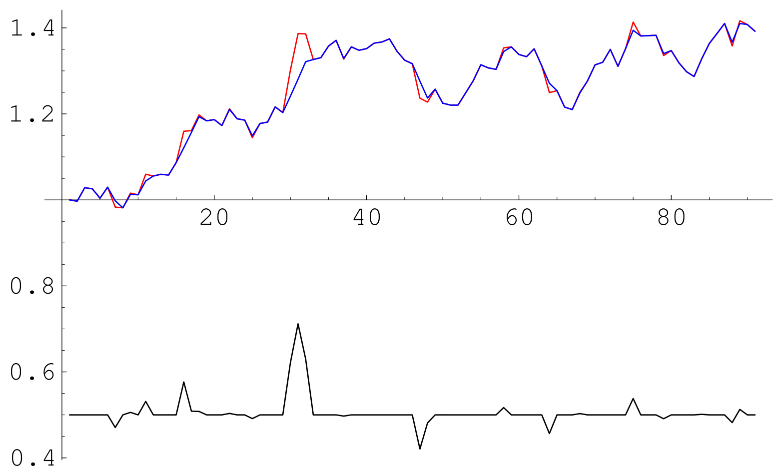

Figure 1: Ideal, exchange prices and delta

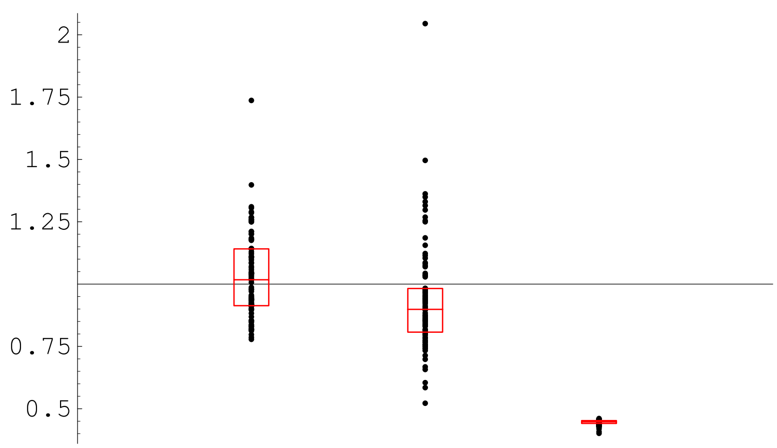

Figure 2: $\kappa/\sigma = 1/2$

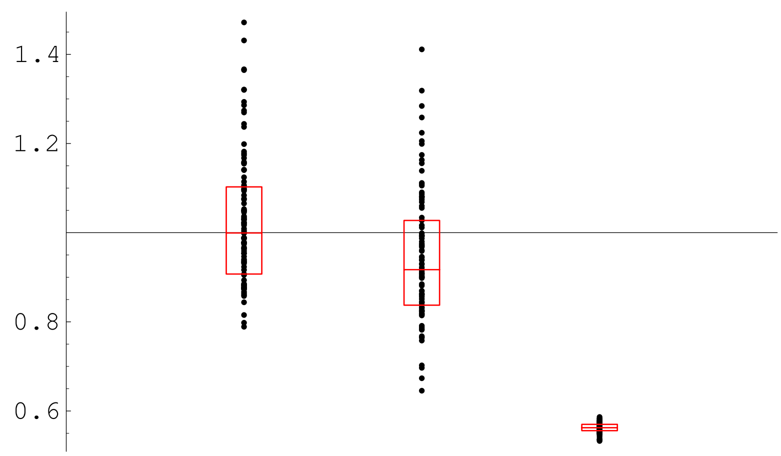

Figure 3: $\kappa/\sigma = 2/3$

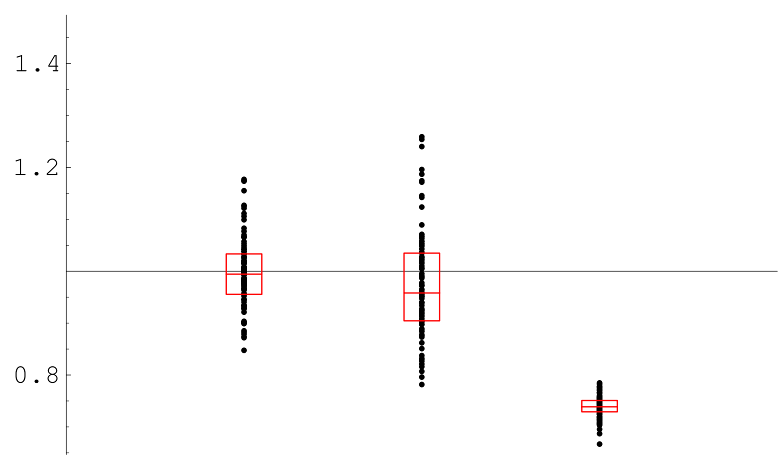

Figure 4: $\kappa/\sigma = 1$

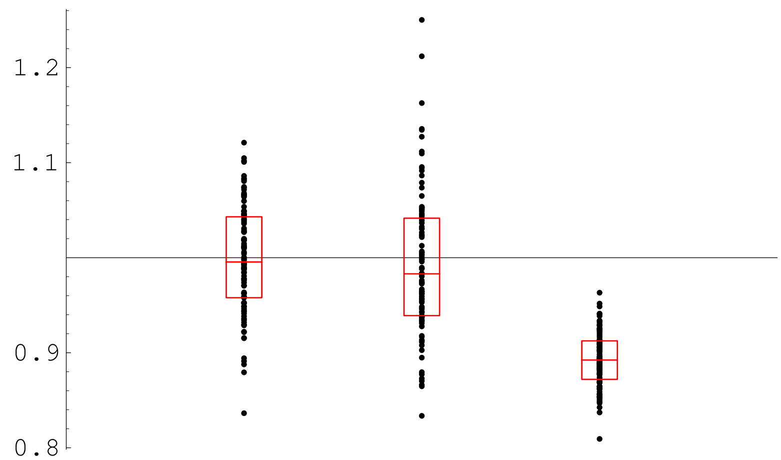

Figure 5: $\kappa/\sigma = 3/2$

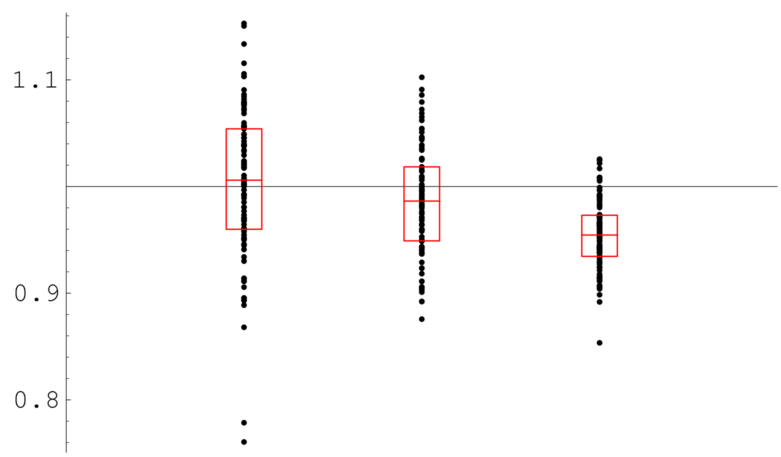

Figure 6: $\kappa/\sigma = 2$