MTH112 – TEST 3 Solutions

Name: ____________________

HONOR CODE: On my honor, I have neither given nor received any aid on this examination.
Signature: ____________________
Note: Show all work on exam in order to receive full credit.

1. Integrate:

(a) $\displaystyle\int_3^7 \frac{1}{(x+1)(x-2)}\,dx$

$$\begin{aligned} A/(x+1) + B/(x-2) &= 1/(x+1)(x-2) \\ A(x-2) + B(x+1) &= 1 \\ x = 2 &\Rightarrow B = 1/3 \\ x = -1 &\Rightarrow A = -1/3 \end{aligned}$$

Hence,

$$\begin{aligned} \int_3^7 \frac{1}{(x+1)(x-2)}\,dx &= (-1/3)(\ln|x+1| - \ln|x-2|)]_3^7 \\ &= \frac{1}{3}\ln\left|\frac{x-2}{x+1}\right|\Big]_3^7 \\ &= \frac{1}{3}(\ln 5/8 - \ln 1/4) = \frac{\ln(5/2)}{3}. \end{aligned}$$

(b) $\displaystyle\int \frac{1}{x(x+1)(x+2)}\,dx$

$$\begin{aligned} A/x + B/(x+1) + C/(x+2) &= 1/x(x+1)(x+2) \\ A(x+1)(x+2) + Bx(x+2) + Cx(x+1) &= 1 \\ x = -1 &\Rightarrow B = -1 \\ x = -2 &\Rightarrow C = 1/2 \\ x = 0 &\Rightarrow A = 1/2 \end{aligned}$$

Hence,

$$\int \frac{1}{x(x+1)(x+2)}\,dx \quad = \quad (1/2)(\ln|x| + \ln|x+2|) - \ln|x+1| + C$$

(c) $\displaystyle\int \frac{x}{(x-3)(x+2)^2}\,dx$

$$\begin{aligned}
B/(x-3) + C/(x+2) + D/(x+2)^2 &= x/((x-3)(x+2)^2) \\
B(x+2)^2 C(x-3)(x+2) + D(x-3) &= x \\
x = 3 &\Rightarrow B = 3/25 \\
x = -2 &\Rightarrow D = 2/5 \\
0x^2 &\Rightarrow C = -B
\end{aligned}$$

Hence,

$$\int \frac{x}{(x-3)(x+2)^2}\,dx = \frac{3}{25}\ln\left|\frac{x-3}{x+2}\right| + \frac{-2}{5(x+2)} + C$$

(d) $\displaystyle\int \frac{5x^2+3x-2}{x^3+2x^2}\,dx$

$$\begin{aligned}
A/x + Bx^2 + C/(x+2) &= (5x^2+3x-2)/(x^3+2x^2) \\
Ax(x+2) + B(x+2) + Cx^2 &= 5x^2+3x-2 \\
x = 0 &\Rightarrow B = -1 \\
x = -2 &\Rightarrow C = 3 \\
5x^2 &\Rightarrow A + C = 5 \Rightarrow A = 3
\end{aligned}$$

Hence,

$$\begin{aligned}
\int \frac{5x^2+3x-2}{x^3+2x^2}\,dx &= \int 2/x\,dx + \int -1/x^2\,dx + 3\int 1/(x+2)\,dx \\
&= 2\ln|x| + (1/x) + 3\ln|x+2| + C
\end{aligned}$$

2. Integrate:

(a) $\int x\sqrt{4-x^2}\,dx$; Let $u = 4 - x^2$, then $du = -2xdx$, so,

$$\begin{aligned}
\int x\sqrt{4-x^2}\,dx &= (-1/2)\int \sqrt{u}du \\
&= (-1/3)u^{3/2} + C \\
&= \frac{-1(4-x^2)^{3/2}}{3} + C
\end{aligned}$$

(b) $\int \sqrt{3-2x-x^2}\,dx$; Complete the square:

$$3 - 2x - x^2 = -(x+1)^2 + 4$$

Let $u = x + 1$, $du = dx$, then use formula 30:

$$\begin{aligned}
\int \sqrt{3-2x-x^2}\,dx &= \int \sqrt{4-u^2}du \\
&= (u/2)\sqrt{4-u^2} + (4/2)\sin^{-1}(u/2) + C \\
&= \frac{(x+1)\sqrt{3-2x-x^2}}{2} + 2\sin^{-1}\frac{x+1}{2} + C
\end{aligned}$$

(c) $\int \frac{\sqrt{9x^2-4}}{x}\,dx$; Let $3x = u$ then use formula 41:

$$\begin{aligned}
\int(\sqrt{9x^2-4})/x\,dx &= \int(\sqrt{u^2-4})/u\,du \\
&= \sqrt{u^2-4} - 2\cos^{-1}(2/u) + C \\
&= \sqrt{9x^2-4} - 2\tan^{-1}\frac{\sqrt{9x^2-4}}{2} + C
\end{aligned}$$

(d) $\int \frac{x}{(x^2+4)^{5/2}}\,dx$; Let $u = x^2 + 4, du = 2xdx$:

$$\begin{aligned}
(1/2)\int 1/u^{5/2}\,du &= (-1/3)\left(1/u^{3/2}\right) + C \\
&= \frac{-1}{3(x^2+4)^{3/2}} + C
\end{aligned}$$

3. Integrate:

(a) $\int \sin x + x^5 \, dx$

$$\begin{aligned} \int \sin x + x^5 \, dx &= \int \sin x \, dx + \int x^5 \, dx \\ &= -\cos x + (1/6)x^6 + C \end{aligned}$$

(b) $\int \frac{1+\sqrt{x+4}}{x} \, dx$

$$\begin{aligned} \int \frac{1+\sqrt{x+4}}{x} \, dx &= \int 1/x \, dx + \int \sqrt{x+4}/x \, dx \\ &= \int 1/x \, dx + \int \frac{x+4}{x\sqrt{x+4}} \, dx \\ &= \int 1/x \, dx + \int \frac{1}{\sqrt{x+4}} \, dx + 4 \int \frac{1}{x\sqrt{x+4}} \, dx \end{aligned}$$

Integrate term by term, using formula 57 for the last integral:

$$\int \frac{1+\sqrt{x+4}}{x} \, dx = \ln|x| + 2\sqrt{x+4} + 2\ln\left|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right| + C$$

(c) $\int \pi \sin x + \cos(1/2)\tan(\pi x) \, dx$

$$\begin{aligned} \int \pi \sin x + \cos(1/2)\tan(\pi x) \, dx &= \pi \int \sin x \, dx + \cos(1/2) \int \tan u \, (du/\pi) \\ &= -\pi \cos x + \cos(1/2) \ln|\sec u|/\pi + C \\ &= -\pi \cos x + \cos(1/2) \ln|\sec(\pi x)|/\pi + C \end{aligned}$$

(d) $\int (1+\sqrt{x})^2 \, dx$

$$\begin{aligned} \int (1+\sqrt{x})^2 \, dx &= \int 1 + 2\sqrt{x} + x \, dx \\ &= x + (4/3)x^{3/2} + x^2/2 + C \end{aligned}$$

4. Integrate:

(a) $\int \frac{1}{x - \sqrt[3]{x}}\, dx$; Use a rationalizing substituion $u = \sqrt[3]{x}, u^3 = x, 3u^2 du = dx$:,

$$\begin{aligned}
\int \frac{3u^2}{u^3 - u}\, du &= 3 \int u/(u^2 - 1)\, du \\
&= 3 \int A/(u-1) + B/(u+1)\, du \\
&= (3/2) \int 1/(u-1) + 1/(u+1)\, du \\
&= (3/2) \ln |u^2 - 1| + C \\
&= \frac{3 \ln |x^{2/3} - 1|}{2} + C
\end{aligned}$$

(b) $\int_1^3 \frac{\sqrt{x-1}}{x+1}\, dx$; Let $u = \sqrt{x-1}, u^2 + 1 = x, 2u\, du = dx$, then substitute $\sqrt{2}w = u$:

$$\begin{aligned}
\int_1^3 \frac{\sqrt{x-1}}{x+1}\, dx &= \int_0^{\sqrt{2}} \frac{u}{u^2+2} 2u\, du \\
&= 2\sqrt{2} \int_0^1 \frac{w^2}{w^2+1}\, dw \\
&= 2\sqrt{2} \int_0^1 1 - 1/(w^2+1)\, dw \\
&= 2\sqrt{2} \left(w - \tan^{-1} w \right)_0^1 \\
&= 2\sqrt{2}(1 - \pi/4)
\end{aligned}$$

(c) $\int \frac{1}{\sqrt{1 + \sqrt{x}}}\, dx$; Let $u = \sqrt{x}, u^2 = x, 2udu = dx$, then use formula 55:

$$\begin{aligned}
\int \frac{1}{\sqrt{1+\sqrt{x}}}\, dx &= 2 \int \frac{u\, du}{\sqrt{1+u}} \\
&= 2((2/3)(u-2)\sqrt{1+u}) + C \\
&= (4/3)(\sqrt{x} - 2)\sqrt{1 + \sqrt{x}} + C
\end{aligned}$$

(d) $\int \frac{\sqrt[3]{x}+1}{\sqrt[3]{x}-1}\,dx$; Use a rationalizing substitution, divide and integrate term by term:

$$\begin{aligned}\int \frac{\sqrt[3]{x}+1}{\sqrt[3]{x}-1}\,dx &= \int \frac{u+1}{u-1} 3u^2\,du \\ &= 3\int u^2+2u+2+2/(u-1)\,du \\ &= u^3+3u^2+6u+6\ln|u-1|+C \\ &= x+3(\sqrt[3]{x})^2+6\sqrt[3]{x}+6\ln|\sqrt[3]{x}-1|+C\end{aligned}$$

5. (a) Factor x^4+1 as the difference of squares by first adding add subtracting the same quantity. Use this factorization to evaluate:

$$\int \frac{1}{x^4+1}\,dx$$

The factorization is

$$x^4+1=(x^2+1)^2-2x^2=(x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1).$$

Use partial fractions, complete the squares, and integrate term by term. I used the computer program *Mathematica* to calculate the solution:

$$\begin{aligned}\int \frac{1}{x^4+1}\,dx &= (2\tan^{-1}(\sqrt{2}x-1)+2\tan^{-1}(\sqrt{2}x+1) \\ &\quad -\ln|1-\sqrt{2}x+x^2|+\ln|1+\sqrt{2}x+x^2|)/(4\sqrt{2})+C\end{aligned}$$

(b) Integrate:

$$\int e^x\cos(3x+4)\,dx$$

Substitute $u=3x+4$, factor out the e^k and use formula 99:

$$\begin{aligned}\int e^x\cos(3x+4)\,dx &= \int e^{(u-4)/3}\cos u\,(du/3) \\ &= (1/3)e^{-4/3}\int e^{u/3}\cos u\,du \\ &= (1/3)e^{-4/3}\left(\frac{e^{u/3}}{(1/3)^2+1}((1/3)\cos u+\sin u)\right)+C \\ &= (3e^x/10)\,((1/3)\cos(3x+4)+\sin(3x+4))+C\end{aligned}$$

(c) Write out the form of the partial fraction decomposition for this function: — do *not* determine the numerical values of the cofficients:

$$\frac{1}{x^6 - x^3}$$

$$\begin{aligned} \frac{1}{x^6 - x^3} &= \frac{1}{x^3(x-1)(x^2+x+1)} \\ &= A/x + B/x^2 + C/x^3 + D/(x-1) + (Ex+F)/(x^2+x+1) \end{aligned}$$

(d) Find the limit:

$$\lim_{x\to\infty} \frac{\ln\ln x}{\sqrt{x}}$$

Use L'Hopital's:

$$\begin{aligned} \lim_{x\to\infty} \frac{\ln\ln x}{\sqrt{x}} &= \lim_{x\to\infty} \frac{(1/\ln x)(1/x)}{1/(-2\sqrt{x})} \\ &= \lim_{x\to\infty} \frac{-2}{\sqrt{x}\ln x} = 0 \end{aligned}$$