Notes on Better Master Theorems for Divide-and-Conquer Recurrences

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Abstract

Techniques for solving divide-and-conquer recurrences are routinely taught to thousands of Computer Science students each year. The dominant approach to solving such recurrences is known as the Master Method [2]. Recently, Akra and Bazzi [1] discovered a surprisingly elegant generalization of the Master Method that yields a very simple formula for solving most divide-and-conquer recurrences. In these notes, we provide a simple inductive proof of the Akra-Bazzi result and we extend the result to handle variations of divide-and-conquer recurrences that commonly arise in practice.

1 Introduction

Divide-and-conquer recurrences are ubiquitous in the analysis of algorithms. Many methods are known for solving recurrences such as

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2T(\lceil n/2 \rceil) + O(n) & \text{if } n > 1, \end{cases}$$

but perhaps the most widely taught approach is the Master Method that is described in the seminal algorithms text by Cormen, Leiserson and Rivest [2].

The Master Method is fairly powerful and results in a closed form solution for divide-and-conquer recurrences with a special (but commonly-occurring) form. Recently Akra and Bazzi [1] discovered a far more general solution to divide-and-conquer recurrences. The Akra-Bazzi analysis is based on a special functional transform that they call the "order transform."

In these notes, we give a simple inductive proof of the Akra-Bazzi result that is suitable for use in an undergraduate algorithms or discrete math class. We also show that the Akra-Bazzi result holds for a more general class of recurrences that commonly arise in practice and that are often considered to be difficult to solve.

2 The Akra-Bazzi Solution

We begin with a simple inductive proof of the Akra-Bazzi result. The result holds for recurrences of the form:

$$T(x) = \begin{cases} \Theta(1) & \text{for } 1 \leq x \leq x_0 \\ \sum_{i=1}^{k} a_i T(b_i x) + g(x) & \text{for } x > x_0 \end{cases} \tag{1}$$

where[1]

1. $x \geq 1$ is a real number,
2. x_0 is a constant such that $x_0 \geq 1/b_i$ and $x_0 \geq 1/(1-b_i)$ for $1 \leq i \leq k$,
3. $a_i > 0$ is a constant for $1 \leq i \leq k$,
4. $b_i \in (0, 1)$ is a constant for $1 \leq i \leq k$,
5. $k \geq 1$ is a constant, and
6. $g(x)$ is a nonnegative function that satisfies the polynomial-growth condition specified below.

Definition. We say that $g(x)$ satisfies the *polynomial-growth condition* if there exist positive constants c_1, c_2 such that for all $x \geq 1$, for all $1 \leq i \leq k$, and for all $u \in [b_i x, x]$,

$$c_1 g(x) \leq g(u) \leq c_2 g(x).$$

Remark. If $|g'(x)|$ is upper bounded by a polynomial in x, then $g(x)$ satisfies the polynomial-growth condition. For example, $g(x) = x^\alpha \log^\beta x$ satisfies the polynomial-growth condition for any constants $\alpha, \beta \in \mathbb{R}$.

Theorem 1 ([1]). *Given a recurrence of the form specified in Equation 1, let p be the unique real number for which $\sum_{i=1}^{k} a_i b_i^p = 1$. Then*

$$T(x) = \Theta\left(x^p \left(1 + \int_1^x \frac{g(u)}{u^{p+1}}\, du\right)\right).$$

Examples.

- If $T(x) = 2T(x/4) + 3T(x/6) + \Theta(x \log x)$, then $p = 1$ and $T(x) = \Theta(x \log^2 x)$.
- If $T(x) = 2T(x/2) + \frac{8}{9}T(3x/4) + \Theta(x^2/\log x)$, then $p = 2$ and $T(x) = \Theta(x^2/\log\log x)$.
- If $T(x) = T(x/2) + \Theta(\log x)$, then $p = 0$ and $T(x) = \Theta(\log^2 x)$.
- If $T(x) = \frac{1}{2}T(x/2) + \Theta(1/x)$, then $p = -1$ and $T(x) = \Theta((\log x)/x)$.
- If $T(x) = 4T(x/2) + \Theta(x)$, then $p = 2$ and $T(x) = \Theta(x^2)$.

[1] These conditions are somewhat less restrictive than those of [1].

The proof of Theorem 1 makes use of the following simple lemma from calculus.

Lemma 1. *If $g(x)$ is a nonnegative function that satisfies the polynomial-growth condition, then there are positive constants c_3, c_4 such that for $1 \le i \le k$ and all $x \ge 1$,*

$$c_3 g(x) \le x^p \int_{b_i x}^{x} \frac{g(u)}{u^{p+1}}\, du \le c_4 g(x).$$

Proof. From the polynomial-growth condition we know that

$$\begin{aligned} x^p \int_{b_i x}^{x} \frac{g(u)}{u^{p+1}}\, du &\le x^p (x - b_i x) \frac{c_2 g(x)}{\min\{(b_i x)^{p+1}, x^{p+1}\}} \\ &= \frac{(1-b_i)c_2}{\min\{1, b_i^{p+1}\}}\, g(x) \\ &\le c_4 g(x) \end{aligned}$$

where we define c_4 to be a constant for which

$$c_4 \ge \frac{(1-b_i)c_2}{\min\{1, b_i^{p+1}\}}$$

for $1 \le i \le k$.

Similarly,

$$\begin{aligned} x^p \int_{b_i x}^{x} \frac{g(u)}{u^{p+1}}\, du &\ge x^p (x - b_i x) \frac{c_1 g(x)}{\max\{(b_i x)^{p+1}, x^{p+1}\}} \\ &= \frac{(1-b_i)c_1}{\max\{1, b_i^{p+1}\}}\, g(x) \\ &\ge c_3 g(x) \end{aligned}$$

where we define c_3 to be a constant for which

$$c_3 \le \frac{(1-b_i)c_2}{\max\{1, b_i^{p+1}\}}$$

for $1 \le i \le k$. $\square$

We will use induction to prove Theorem 1, and so it will be helpful to partition the domain of x into intervals $I_0 = [1, x_0]$ and $I_j = (x_0 + j - 1, x_0 + j]$ for $j \ge 1$.

By the definition of x_0, we know that if $x \in I_j$ for some $j \ge 1$, then for $1 \le i \le k$, $b_i x \in I_{j'}$ for some $j' < j$. This is because $b_i x > b_i(x_0 + j - 1) \ge b_i x_0 \ge 1$, and because $b_i x \le b_i(x_0 + j) \le x_0 + j - (1 - b_i)x_0 \le x_0 + j - 1$. As a consequence, we know that the value of T in any interval after $[1, x_0]$ depends only on the values of T in prior intervals.

Proof of Theorem 1. We first show that there is a positive constant c_5 such that for all $x > x_0$,

$$T(x) \geq c_5 x^p \left(1 + \int_1^x \frac{g(u)}{u^{p+1}}\, du\right).$$

The proof is by induction on the interval I_j containing x. The base case when $j = 0$ follows from the fact that $T(x) = \Theta(1)$ when $x \in [1, x_0]$ (provided that we choose c_5 small enough).

The inductive step is argued as follows:

$$\begin{aligned}
T(x) &= \sum_{i=1}^{k} a_i T(b_i x) + g(x) \\
&\geq \sum_{i=1}^{k} a_i c_5 (b_i x)^p \left(1 + \int_1^{b_i x} \frac{g(u)}{u^{p+1}}\, du\right) + g(x) && \text{(by induction)} \\
&= c_5 x^p \sum_{i=1}^{k} a_i b_i^p \left(1 + \int_1^x \frac{g(u)}{u^{p+1}}\, du - \int_{b_i x}^x \frac{g(u)}{u^{p+1}}\, du\right) + g(x) \\
&\geq c_5 x^p \sum_{i=1}^{k} a_i b_i^p \left(1 + \int_1^x \frac{g(u)}{u^{p+1}}\, du - \frac{c_4}{x^p}\, g(x)\right) + g(x) && \text{(by Lemma 1)} \\
&= c_5 x^p \left(1 + \int_1^x \frac{g(u)}{u^{p+1}}\, du - \frac{c_4}{x^p}\, g(x)\right) + g(x) \\
&= c_5 x^p \left(1 + \int_1^x \frac{g(u)}{u^{p+1}}\, du\right) + g(x) - c_5 c_4 g(x) \\
&\geq c_5 x^p \left(1 + \int_1^x \frac{g(u)}{u^{p+1}}\, du\right)
\end{aligned}$$

provided that $c_5 \leq 1/c_4$.

The proof that there is a positive constant c_6 such that for all $x > x_0$,

$$T(x) \leq c_6 x^p \left(1 + \int_1^x \frac{g(u)}{u^{p+1}}\, du\right)$$

is nearly identical. We need only insure that c_6 is chosen large enough so that the base case is satisfied and so that $c_6 \geq 1/c_3$. As a consequence, we can conclude that

$$T(x) = \Theta\left(x^p \left(1 + \int_1^x \frac{g(u)}{u^{p+1}}\, du\right)\right),$$

as claimed. $\square$

Remark. If $g(x)$ grows faster than any polynomial in x, then $T(x) = \Theta(g(x))$. Hence, Theorem 1 does not necessarily hold if $g(x)$ does not satisfy the polynomial-growth condition.

3 Variations

Although the class of recurrences analyized in Section 2 is quite broad, recurrences that arise in practice often differ in small ways from the class specified in Equation 1. For example, in algorithm design, recurrences of the form

$$T(x) \leq \sum_{i=1}^{k} a_i T(\lceil b_i x \rceil) + g(x)$$

are common.

Generally speaking, the inclusion of floors and ceilings in a recurrence does not significantly change the nature of the solution (e.g., see [1, 2]), but the proofs of this fact tend to be fairly tedious and specialized in nature. In what follows, we describe a general class of variations (which includes floors and ceilings) and we show that the variations in this class do not affect the solution of the recurrence (up to constant factors). In particular, we show that the solution of Theorem 1 holds for all recurrences of the form:

$$T(x) = \begin{cases} \Theta(1) & \text{for } 1 \leq x \leq x_0 \\ \sum_{i=1}^{k} a_i T(b_i x + h_i(x)) + g(x) & \text{for } x > x_0 \end{cases} \tag{2}$$

where

1. x, x_0, a_i, b_i, k, and $g(x)$ all satisfy the conditions specified in Section 2,
2. there is some constant $\epsilon > 0$ for which $|h_i(x)| \leq x/(\log^{1+\epsilon} x)$ for $1 \leq i \leq k$ whenever $x \geq x_0$,
3. there exist positive constants c_1 and c_2 such that for all $x \geq 1$, for all $1 \leq i \leq k$, and for all $u \in [b_i x + h_i(x), x]$,

   $$c_1 g(x) \leq g(u) \leq c_2 g(x),$$

   and
4. x_0 is chosen to be a large enough constant[2] so that for any $i \leq k$ and any $x \geq x_0$,

   (a) $\left(1 - \frac{1}{b_i \log^{1+\epsilon} x}\right)^p \left(1 + \frac{1}{\log^{\epsilon/2}\left(b_i x + \frac{x}{\log^{1+\epsilon} x}\right)}\right) \geq 1 + \frac{1}{\log^{\epsilon/2} x}$,

   (b) $\left(1 + \frac{1}{b_i \log^{1+\epsilon} x}\right)^p \left(1 - \frac{1}{\log^{\epsilon/2}\left(b_i x + \frac{x}{\log^{1+\epsilon} x}\right)}\right) \leq 1 - \frac{1}{\log^{\epsilon/2} x}$,

   (c) $\frac{1}{2}\left(1 + \frac{1}{\log^{\epsilon/2} x}\right) \leq 1$,

   (d) $2\left(1 - \frac{1}{\log^{\epsilon/2} x}\right) \geq 1$.

[2]Such a constant value of x_0 can be shown to exist using standard Taylor series expansions and asymptotic analysis.

For example, we might choose h_i so that

$$h_i(x) = \lceil b_i x \rceil - b_i x,$$

thereby extending Theorem 1 to handle ceiling functions. In this case, $|h_i(x)| < 1$. We can also use much larger functions, however. For example, we could set $h_i(x) = -\sqrt{x}$ or $h_i(x) = x/(\log^2 x)$ for $x > 1$.

To analyze the more general recurrence, we will need the following analogue of Lemma 1.

Lemma 2. *There are positive constants c_3, c_4 such that for $1 \le i \le k$ and all $x \ge 1$,*

$$c_3 g(x) \le x^p \int_{b_i x + h_i(x)}^{x} \frac{g(u)}{u^{p+1}}\, du \le c_4 g(x).$$

Proof. The proof is identical to that for Lemma 1 except that we use constraint 3 above in place of the polynomial-growth condition of Section 2. □

Theorem 2. *Given a recurrence of the form specified in Equation 2, let p be the unique real number for which $\sum_{i=1}^{k} a_i b_i^p = 1$. Then*

$$T(x) = \Theta\left(x^p \left(1 + \int_1^x \frac{g(u)}{u^{p+1}}\, du\right)\right).$$

Proof. The proof is very similar to that of Theorem 1. The main difference is that we use a slightly stronger inductive hypothesis. In particular, we begin by showing that there is a positive constant c_5 such that for all $x > x_0$,

$$T(x) \ge c_5 x^p \left(1 + \frac{1}{\log^{\epsilon/2} x}\right) \left(1 + \int_1^x \frac{g(u)}{u^{p+1}}\, du\right).$$

The proof is by induction on the interval I_j containing x. The base case when $j = 0$ follows from the fact that $T(x) = \Theta(1)$ when $x \in [1, x_0]$ (provided that c_5 is chosen to be a small enough constant).

The inductive step is argued as follows:

$$
\begin{aligned}
T(x) &= \sum_{i=1}^{k} a_i T(b_i x + h_i(x)) + g(x) \\
&\geq \sum_{i=1}^{k} a_i c_5 (b_i x + h_i(x))^p \left(1 + \frac{1}{\log^{\epsilon/2}(b_i x + h_i(x))}\right) \\
&\qquad \times \left(1 + \int_1^{b_i x + h_i(x)} \frac{g(u)}{u^{p+1}}\, du\right) + g(x) \qquad \text{(by induction)} \\
&\geq \sum_{i=1}^{k} a_i b_i^p c_5 x^p \left(1 - \frac{1}{b_i \log^{1+\epsilon} x}\right)^p \left(1 + \frac{1}{\log^{\epsilon/2}\left(b_i x + \frac{x}{\log^{1+\epsilon} x}\right)}\right) \\
&\qquad \times \left(1 + \int_1^{x} \frac{g(u)}{u^{p+1}}\, du - \int_{b_i x + h_i(x)}^{x} \frac{g(u)}{u^{p+1}}\, du\right) + g(x) \qquad \text{(by the bounds on } h) \\
&\geq \sum_{i=1}^{k} a_i b_i^p c_5 x^p \left(1 + \frac{1}{\log^{\epsilon/2} x}\right) \left(1 + \int_1^{x} \frac{g(u)}{u^{p+1}}\, du - \frac{c_4}{x^p}\, g(x)\right) + g(x) \\
&\qquad \text{(by constraint 4(a) on } x_0 \text{ and Lemma 2)} \\
&= c_5 x^p \left(1 + \frac{1}{\log^{\epsilon/2} x}\right) \left(1 + \int_1^{x} \frac{g(u)}{u^{p+1}}\, du - \frac{c_4}{x^p}\, g(x)\right) + g(x) \\
&= c_5 x^p \left(1 + \frac{1}{\log^{\epsilon/2} x}\right) \left(1 + \int_1^{x} \frac{g(u)}{u^{p+1}}\, du\right) + g(x) - c_5 c_4 \left(1 + \frac{1}{\log^{\epsilon/2} x}\right) g(x) \\
&\geq c_5 x^p \left(1 + \frac{1}{\log^{\epsilon/2} x}\right) \left(1 + \int_1^{x} \frac{g(u)}{u^{p+1}}\, du\right)
\end{aligned}
$$

provided that $c_5 \leq 1/(2c_4)$ (by constraint 4(c) on x_0).

The proof of the upper bound is quite similar. In this case, we show by induction that there is a positive constant c_6 such that for all $x > x_0$,

$$
T(x) \leq c_6 x^p \left(1 - \frac{1}{\log^{\epsilon/2} x}\right) \left(1 + \int_1^{x} \frac{g(u)}{u^{p+1}}\, du\right).
$$

The base case is as before. The inductive step is argued as follows:

$$
\begin{aligned}
T(x) &= \sum_{i=1}^{k} a_i T(b_i x + h_i(x)) + g(x) \\
&\le \sum_{i=1}^{k} a_i c_6 (b_i x + h_i(x))^p \left(1 - \frac{1}{\log^{\epsilon/2}(b_i x + h_i(x))}\right) \\
&\qquad\qquad \times \left(1 + \int_1^{b_i x + h_i(x)} \frac{g(u)}{u^{p+1}}\, du\right) + g(x) \qquad\qquad \text{(by induction)} \\
&\le \sum_{i=1}^{k} a_i b_i^p c_6 x^p \left(1 + \frac{1}{b_i \log^{1+\epsilon} x}\right)^p \left(1 - \frac{1}{\log^{\epsilon/2}\left(b_i x + \frac{x}{\log^{1+\epsilon} x}\right)}\right) \\
&\qquad\qquad \times \left(1 + \int_1^{x} \frac{g(u)}{u^{p+1}}\, du - \int_{b_i x + h_i(x)}^{x} \frac{g(u)}{u^{p+1}}\, du\right) + g(x) \qquad \text{(by the bounds on } h) \\
&\le \sum_{i=1}^{k} a_i b_i^p c_6 x^p \left(1 - \frac{1}{\log^{\epsilon/2} x}\right) \left(1 + \int_1^{x} \frac{g(u)}{u^{p+1}}\, du - \frac{c_3}{x^p} g(x)\right) + g(x) \\
&\qquad\qquad \text{(by constraint 4(b) on } x_0 \text{ and Lemma 2)} \\
&= c_6 x^p \left(1 - \frac{1}{\log^{\epsilon/2} x}\right) \left(1 + \int_1^{x} \frac{g(u)}{u^{p+1}}\, du - \frac{c_3}{x^p} g(x)\right) + g(x) \\
&= c_6 x^p \left(1 - \frac{1}{\log^{\epsilon/2} x}\right) \left(1 + \int_1^{x} \frac{g(u)}{u^{p+1}}\, du\right) + g(x) - c_3 c_6 \left(1 - \frac{1}{\log^{\epsilon/2} x}\right) g(x) \\
&\le c_6 x^p \left(1 - \frac{1}{\log^{\epsilon/2} x}\right) \left(1 + \int_1^{x} \frac{g(u)}{u^{p+1}}\, du\right)
\end{aligned}
$$

provided that $c_6 \ge 2/c_3$ (by constraint 4(d) on x_0).

Hence, we can conclude that

$$T(x) = \Theta\left(x^p \left(1 + \int_1^{x} \frac{g(u)}{u^{p+1}}\, du\right)\right),$$

as desired. $\square$

Remark. It is worth noting that the $x/\log^{1+\epsilon} x$ limit on the size of $|h_i(x)|$ is nearly tight, since the solution of the recurrence

$$T(x) = \begin{cases} \Theta(1) & \text{for } 1 \le x \le x_0 \\ 2T\left(\frac{x}{2} + \frac{x}{\log x}\right) & \text{for } x > x_0 \end{cases}$$

is $T(x) = x \log^{\Theta(1)} x$, which is different than the solution of $\Theta(x)$ for the recurrence without the $x/\log x$ term.

References

[1] M. Akra and L. Bazzi. "On the solution of linear recurrence equations." To appear, 1996.

[2] Thomas H. Cormen, Charles E. Leierson, and Ronald L. Rivest. *Introduction to Algorithms.* The MIT Press, Cambridge, Massachusetts, 1990.