Data Structures and Algorithm Analysis (CSC317)

"We already have quite a few people who know how to divide. So essentially, we're now looking for people who know how to conquer."

Divide and conquer
From previous class

Go over proofs for growth of functions (on the board)
Goals

What kind of recurrences arise in algorithms and how do we solve more generally (than what we saw for merge sort)?

• More recurrence examples

• Run time not always intuitive, so need tools
Usefulness in recent applications

Application of divide-and-conquer algorithm paradigm to improve the detection speed of high interaction client honeypots. Seifert et al. 2008. “…one needs to be able to find malicious servers on a network… Client honeypots are the new emerging technology that can perform such searches… they are faced with crawling the Internet with its millions of servers. Finding a malicious server might be similar to finding a needle in a haystack.”
Usefulness in recent applications


- Sequential: Make server request one by one to a large set of servers, and detect malicious

- Detect malicious by making server requests in parallel for set of servers. Each time mark a given set as malicious or not, but can’t determine server identity without manual check

- Use the divide and conquer approach
Usefulness in recent applications


Figure 7: Divide-and-conquer Algorithm Example
More Detailed Algorithm Examples
Max Subarray Problem

**Problem**: Can buy stock once, sell stock once. Want to maximize profit; allowed to look into the future
Max Subarray Problem

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**Problem:** Can buy stock once, sell stock once. Want to maximize profit; allowed to look into the future.

**Complexity?**
Max Subarray Problem

**Brute force:** Try every possible pair of buy and sell dates:

\[
\binom{n}{2} = \frac{n!}{(n-2)!2!} = \frac{n(n-1)(n-2)!}{(n-2)!2!} = \frac{n(n-1)}{2} = \Theta(n^2)
\]
Max Subarray Problem

**Brute force:** Try every possible pair of buy and sell dates:

\[
\binom{n}{2} = \frac{n!}{(n-2)!2!} = \frac{n(n-1)(n-2)!}{(n-2)!2!} = \frac{n(n-1)}{2} = \Theta(n^2)
\]

Can we do better?
Max Subarray Problem

Brute force: Can we do better? Try to reframe as greatest sum of any contiguous array

Best contiguous sum representing gain from buy to sell!
Max Subarray Problem

**Brute force:** Can we do better? Try to reframe as greatest sum of any contiguous array
Max Subarray Problem

Try to reframe as greatest sum of any contiguous Array. Efficiency?

![Graph showing stock price over 17 days]

<table>
<thead>
<tr>
<th>Day</th>
<th>Price</th>
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<tbody>
<tr>
<td>1</td>
<td>13</td>
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The maximum subarray problem is interesting only when the array contains some negative numbers. If all the array entries were nonnegative, then the maximum-subarray problem would present no challenge, since the entire array would give the greatest sum.

Therefore, a maximum subarray of \( A_{i:j} \) must have the greatest maximum sum.

As a solution using divide-and-conquer, suppose we want to find a maximum subarray of the subarray \( A_{i:j} \), that although computing the cost of one subarray might take time proportional to the length of the subarray, when computing all subarray sums, we can organize the computation so that each subarray sum takes time.

Let's think about how we might solve the maximum-subarray problem using the divide-and-conquer technique. Suppose we want to find a maximum subarray of the subarray \( A_{i:j} \). We can find maximum subarrays of sum over all subarrays entirely in the subarray, or crossing the midpoint. We can find maximum subarrays of

Thus, all that is left to do is find a maximum subarray of the subarray

Figure 4.3
Max Subarray Problem

Try to reframe as greatest sum of any contiguous array. **Efficiency? It’s still brute force**
Max Subarray Problem

Try to reframe as greatest sum of any contiguous array. **If all the array values were positive?**
Max Subarray Problem

Try to reframe as greatest sum of any contiguous array. If all the array values were positive?

A = [1 10 12 13 23 33 2]
Max Subarray Problem

Try to reframe as greatest sum of any contiguous array. **If all the array values were positive?**

\[ A = [1 \ 10 \ 12 \ 13 \ 23 \ 33 \ 2] \]

Not interesting – summing all array values gives the max…
Max Subarray Problem

For positive and negative values, it’s still brute force. Divide and Conquer?

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Max Subarray Problem

Divide and conquer approach
Where can max subarray be?

low

middle

high
Max Subarray Problem

Divide and conquer approach
Where can max subarray be?
Max Subarray Problem

Divide and conquer approach

1. Divide subarray into two equal size subarrays, A[low..mid] and A[mid+1..high]
2. Conquer, finding max of subarrays A[low..mid] and A[mid+1..high]
3. Combine, finding best solution of:
   a. the two solutions found in conquer step
   b. solution of subarray crossing the midpoint
Max Subarray Problem

Divide and conquer approach

Keep recursing until low=high (one element left)-

1. Divide subarray into two equal size subarrays, A[low..mid] and A[mid+1..high]
2. Conquer, finding max of subarrays A[low..mid] and A[mid+1..high]
3. Combine, finding best solution of:
   a. the two solutions found in conquer step
   b. solution of subarray crossing the midpoint
Max Subarray Problem

Algorithm for max subarray crossing midpoint?
Max Subarray Problem

Subarray crossing midpoint

[-16 -23 18 20 | -7 12 -5 -22]

- Start from middle
- Traverse to left until get maximum sum (?)
- Traverse to right until get maximum sum (?)
- Return total left and right sum (?)

Complexity?
Max Subarray Problem

Subarray crossing midpoint

[-16 -23 18 20 | -7 12 -5 -22]

- Start from middle
- Traverse to left until get maximum sum (?)
- Traverse to right until get maximum sum (?)
- Return total left and right sum (?)

Complexity? \( \Theta(n) \)
Max Subarray Problem

Divide and conquer approach: full example:

[-16 -23 18 20 -7 12 -5 -22]

On the board…
Max Subarray Problem

Divide and conquer approach: example:

\[-16 \ -23 \ \boxed{18\ 20\ -7\ 12\ -5\ -22}\]

On the board…
Max Subarray Problem

Costs:

1. Divide: $\Theta(1)$

2. Conquer: $2T\left(\frac{n}{2}\right)$

3. Combine: $\Theta(n) + \Theta(1)$

Subarray crossing Comparisons
Max Subarray Problem

Costs:

1. Divide: $\Theta(1)$

2. Conquer: $2T\left(\frac{n}{2}\right)$

3. Combine: $\Theta(n) + \Theta(1)$

Subarray crossing

Comparisons

Total: $T(n) = 2T\left(\frac{n}{2}\right) + \Theta(n) = ?$
Max Subarray Problem

Costs:

1. Divide: $\Theta(1)$

2. Conquer: $2T(\frac{n}{2})$

3. Combine: $\Theta(n) + \Theta(1)$
   - Subarray crossing
   - Comparisons

Total: $T(n) = 2T(\frac{n}{2}) + \Theta(n) = \Theta(n \log n)$
   - Like merge sort....
Classical example: matrix multiplication

\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} \\
a_{21} & a_{22} & a_{23} & a_{24} \\
a_{31} & a_{32} & a_{33} & a_{34} \\
a_{41} & a_{42} & a_{43} & a_{44}
\end{bmatrix}
\begin{bmatrix}
b_{11} & b_{12} & b_{13} & b_{14} \\
b_{21} & b_{22} & b_{23} & b_{24} \\
b_{31} & b_{32} & b_{33} & b_{34} \\
b_{41} & b_{42} & b_{43} & b_{44}
\end{bmatrix} =
\begin{bmatrix}
c_{11} & c_{12} & c_{13} & c_{14} \\
c_{21} & c_{22} & c_{23} & c_{24} \\
c_{31} & c_{32} & c_{33} & c_{34} \\
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  c_{41} & c_{42} & c_{43} & c_{44}
\end{bmatrix}
\]

\[
c_{11} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} + a_{14}b_{41};
\]

...
Classical example: matrix multiplication

\[
\begin{bmatrix}
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a_{21} & a_{22} & a_{23} & a_{24} \\
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\end{bmatrix}
\]

\(c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}\)

- Run time?
Classical example: matrix multiplication

\[
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} & a_{14} \\
  a_{21} & a_{22} & a_{23} & a_{24} \\
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\end{bmatrix}
\]

\[c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}\]

- Run time? \( O(n^3) \)

Answer: Naïve implementation
Classical example: matrix multiplication

Square-Matrix-Multiply(A,B)

1. n = A.rows
2. Let C be a new n by n matrix
3. For i=1 to n
4. For j=1 to n
5. cij = 0
6. For k=1 to n
7. cij = cij + aik bkj
8. Return C

$O(n^3)$
Classical example: matrix multiplication

• Run time?

Answer: Naïve implementation $O(n^3)$

Can we do better? (next class; divide and conquer approaches)