14

SIAM J. COMPUT.
Vol. 23, No. 3, pp. 553–562, June 1994

PACKET TRANSMISSION IN A NOISY-CHANNEL RING NETWORK*

VICTOR PESTIEN†, S. RAMAKRISHNAN†, AND DILIP SARKAR†

Abstract. Assume that n stations, each with a buffer to hold only one packet at a time, are connected as a ring and that data packets are transmitted counterclockwise. A station will attempt to transmit a packet to the next station only if (i) it has a packet to send and (ii) the next station's buffer is empty. The communication channels connecting the stations are noisy, and there is a fixed probability p ($0 < p < 1$) of error-free transmission of a packet from one station to the next in one attempt.

An exact expression for the long-run average time for a packet to go around the ring is derived. (A special case of this answers a question raised by Berman and Simon in [*Proc. 20th ACM Symp. on Theory of Computing*, ACM Press, 1988, pp. 66–77].) For fixed n and p, the throughput of the system is maximum when the number of packets is an integer closest to $n/2$.

Key words. ring network, packet communication, equilibrium distribution, Markov chain

AMS subject classifications. primary: 68M10; secondary: 60J10, 94A05

1. Introduction. Local area networks are a subject of increasing importance for efficient data communication. A ring network is a particularly attractive local area network, since its architecture uses simple control software and interfaces. In a unidirectional ring network, the host computers are connected to the network via ring interface hardware. Each interface is called a station in the ring. A station transmits its message to the next station in the ring, and the message circulates around the network until it reaches the destination station where it is forwarded to the host computer. The interface hardware can identify messages intended for its host. A host may transmit its packet by passing it to its attached station using one of several standard protocols (see Tanenbaum [6]).

In this paper, we study the asymptotic average rate of transmission of packets of messages in a ring network. We assume that a station can store only one packet of a message at a time in its buffer. Thus, a station can transmit a packet to the next station if the next station has an empty buffer. During transmission, a packet can become corrupted in the channel. In other words, the transmission channel is not noise-free. Assume that there is a probability p of error-free transmission in one attempt. If a packet is corrupted during transmission, it must be retransmitted. Also assume that all the stations synchronously attempt to transmit packets.

We assume that there is a fixed interval of time between consecutive transmission attempts. Within this interval, each station will determine the status of the buffer of the next station, attempt transmission, and receive acknowledgment. We further assume that the transmission of the acknowledgment is error-free. This assumption is harmless since the size of the acknowledgment packet is very small compared to the size of a packet of information, and hence it can be encoded using some error-correcting code.

This paper illustrates how the theory of Markov chains could be naturally used to model and analyze a computing problem. In §2 the system is formalized as a Markov chain and its equilibrium distribution is obtained. This distribution is then used in §3 to calculate the asymptotic average cycle time of a packet. In §4, the condition for maximizing throughput is established. Throughput of the system is the expected number of packets successfully transmitted from one station to the next in one unit of time. Section 5 gives some concluding remarks and discusses connections with the work of Berman and Simon [1].

2. Stationary probability distribution. Consider n *stations* arranged in a circle. We assume that the stations are numbered $0, 1, \ldots, n-1$. Also, for any integer j, by *station* j, we

*Received by the editors July 12, 1989; accepted for publication (in revised form) January 19, 1993.

†Department of Mathematics and Computer Science, University of Miami, Coral Gables, Florida 33124-4250.

shall mean the station $j \bmod n$. Suppose there are k packets ($k < n$) that are to be transmitted counterclockwise around the circle under the following conditions:

(i) packets can move only at discrete times $t = 1, 2, \ldots$;

(ii) at every time instant t, each packet is at some station and there is at most one packet at each station;

(iii) at every time instant t and for every integer j, a packet located at station j can move only if station $(j+1)$ does not have a packet; if it *can* move, it *will* move, independently of other packets that can move, with probability p ($0 < p < 1$) to station $(j+1)$, and it will stay at station j with probability $(1-p)$.

We want to calculate the long-run average speed with which a packet goes around the circle. We shall formalize this stochastic process as a discrete-time Markov chain with stationary transition probabilities. (All the definitions and results from the theory of Markov chains that we shall need can be found in Chung [2] or Feller [3].)

Given a time instant, we say that the *local state* of a station is

$$\begin{cases} 1 & \text{if it has a packet at that instant,} \\ 0 & \text{if it does not.} \end{cases}$$

The state space S of our Markov chain consists of n-tuples $s = (s_0, \ldots, s_{n-1})$, where s_j is the local state of station j. Formally,

$$S = \left\{ s \in \{0, 1\}^n : \text{ exactly } k \text{ coordinates of } s \text{ are } 1 \right\}.$$

For $s \in S$ and for any integer j, by s_j we shall mean the ($j \bmod n$)th coordinate of s. For $s \in S$ and any integer j, we say that j is an *opportunity station* of state s if $s_j s_{j+1} = 10$; and we say that j is a *post-opportunity station* of state s if $s_j s_{j+1} = 01$. An opportunity station j is one that has a packet that can move in a one-step transition to station $j+1$ since there is no packet at station $j+1$, while a post-opportunity station is one from which a packet may have just moved to station $j+1$. Let

$$\mathrm{OPP}(s) = \{ j : \ 0 \le j \le n-1 \ \text{ and } j \text{ is an opportunity station of state } s \ \}$$

and

$$\mathrm{POSTOPP}(s) = \{ j : \ 0 \le j \le n-1 \ \text{ and } j \text{ is a post-opportunity station of state } s \ \}.$$

The states that can result from a state s after a one-step transition are in one-to-one correspondence with subsets of $\mathrm{OPP}(s)$, the subset identifying the stations from which a packet actually moves. If $A \subseteq \mathrm{OPP}(s)$, the state s^A that will result from s when A is the set of stations from which packets have just moved is described formally as follows:

(i) for $j \in A$, $s_j^A s_{j+1}^A = 01$;

(ii) if s_j^A is not specified by (i), then $s_j^A = s_j$.

The entries of the transition matrix P of our Markov chain are defined by

$$P(s, s^A) = p^{|A|}(1-p)^{|\mathrm{OPP}(s)|-|A|}, \tag{2.1}$$

for $s \in S$ and $A \subseteq \mathrm{OPP}(s)$. ($|\cdot|$ denotes cardinality.) All other entries of the transition matrix are zero.

Thus for each $s \in S$, $P(s, s')$ is nonzero for exactly $2^{|\mathrm{OPP}(s)|}$ states s'. However, given states $s, s' \in S$, it can easily be verified that in $2n$ one-step transitions, s can be transformed into s' with positive probability. Thus the Markov chain with transition matrix P is irreducible. It is also positive recurrent, because the set S is finite. Therefore, as is well known, there is a unique stationary probability distribution for P, i.e., there is a unique vector π on S such that

$$\pi(s) = \sum_{s' \in S} \pi(s') P(s', s) \quad \text{for all } s \in S, \tag{2.2}$$

$$\pi(s) > 0 \quad \text{for all } s \in S, \tag{2.3}$$

and

$$\sum_{s \in S} \pi(s) = 1. \tag{2.4}$$

The stationary probability distribution of P can thus be obtained by first getting a vector that satisfies (2.3) and (2.2) and then normalizing it.

Let γ be the vector on S defined by

$$\gamma(s) = (1-p)^{-|\mathrm{OPP}(s)|}. \tag{2.5}$$

THEOREM 2.1. *The vector* γ *defined by* (2.5) *satisfies* (2.3) *and* (2.2) *for the matrix* P *defined by* (2.1).

Proof. Plainly, since $0 < p < 1$, γ satisfies (2.3). To verify (2.2), given $s \in S$, by the definition of P, the summation on the right side of (2.2) need be taken only over those $s' \in S$ such that for some $A \subseteq \mathrm{OPP}(s)$, $s = (s')^A$. In such a case, $A \subseteq \mathrm{POSTOPP}(s)$. Moreover, given $s \in S$ and $A \subseteq \mathrm{POSTOPP}(s)$, there is a unique state s' satisfying $(s')^A = s$. We shall denote this state by s^{-A}. With this notation (2.2) can be rewritten as follows:

$$\gamma(s) = \sum_{A \subseteq \mathrm{POSTOPP}(s)} \gamma(s^{-A}) P(s^{-A}, s) \quad \text{for all } s \in S. \tag{2.6}$$

The proof of (2.6) will use the following two lemmas.

LEMMA 2.2. *If* $s \in S$, *then*

$$|\mathrm{OPP}(s)| = |\mathrm{POSTOPP}(s)|.$$

Proof. For $j \in \mathrm{OPP}(s)$, let j' be the largest integer such that $s_i = 0$ for $j < i \le j'$ and let $j'' = j' \bmod n$. It is easy to see that the pairing of j and j'' gives a one-to-one correspondence between $\mathrm{OPP}(s)$ and $\mathrm{POSTOPP}(s)$. □

LEMMA 2.3. *If* $s \in S$ *and* $A \subseteq \mathrm{POSTOPP}(s)$, *then*

$$\gamma(s^{-A})\ P(s^{-A}, s) = \left(\frac{p}{1-p}\right)^{|A|}.$$

Proof. Since $(s^{-A})^A = s$ by definition, we have

$$\begin{aligned}\gamma(s^{-A})\ P(s^{-A}, s) &= (1-p)^{-|\mathrm{OPP}(s^{-A})|} \cdot p^{|A|} \cdot (1-p)^{|\mathrm{OPP}(s^{-A})|-|A|} \\ &= \left(\frac{p}{1-p}\right)^{|A|}. \qquad \square\end{aligned}$$

We now finish the proof of Theorem 2.1 by verifying (2.6). Note that

$$
\begin{aligned}
&\sum_{A \subseteq \mathrm{POSTOPP}(s)} \gamma(s^{-A}) \cdot P(s^{-A}, s) \\
&\quad = \sum_{A \subseteq \mathrm{POSTOPP}(s)} \left(\frac{p}{1-p}\right)^{|A|} \qquad \text{[by Lemma 2.3]} \\
&\quad = \sum_{\ell=0}^{|\mathrm{POSTOPP}(s)|} \binom{|\mathrm{POSTOPP}(s)|}{\ell} \left(\frac{p}{1-p}\right)^{\ell} \\
&\quad = \left(1 + \frac{p}{1-p}\right)^{|\mathrm{POSTOPP}(s)|} \qquad \text{[by the binomial theorem]} \\
&\quad = (1-p)^{-|\mathrm{POSTOPP}(s)|} \\
&\quad = (1-p)^{-|\mathrm{OPP}(s)|} \qquad \text{[by Lemma 2.2].}
\end{aligned}
$$

The proof of Theorem 2.1 is thus complete. □

COROLLARY 2.4. *If π is defined by*

$$\pi(s) = \frac{\gamma(s)}{\sum_{s' \in S} \gamma(s')}, \tag{2.7}$$

where γ satisfies (2.5), *then π is the unique stationary probability distribution of the Markov chain with transition probability matrix P.*

3. Average cycle time. A natural measure of the progress of the system is the number of packet movements. In a one-step transition from state s to state s', this is the number of opportunity stations of s that successfully transmit a packet and hence are post-opportunity stations of s', i.e.,

$$|\mathrm{OPP}(s) \cap \mathrm{POSTOPP}(s')|.$$

Of course, such a transition would have positive probability only when $s' = s^A$ for some $A \subseteq \mathrm{OPP}(s)$; and in such a case $|\mathrm{OPP}(s) \cap \mathrm{POSTOPP}(s')| = |A|$. If $s^0, s^1, \ldots, s^t, \ldots$ denote the sequence of states generated by our Markov chain (s^t is the state at time instant t), the long-run average time per unit progress is

$$\lim_{t \to \infty} \frac{t}{\sum_{i=1}^{t} |\mathrm{OPP}(s^{i-1}) \cap \mathrm{POSTOPP}(s^i)|}. \tag{3.1}$$

Using the strong law of large numbers for Markov chains, one can show (see below) that with probability one,

$$\lim_{t \to \infty} \frac{t}{\sum_{i=1}^{t} |\mathrm{OPP}(s^{i-1}) \cap \mathrm{POSTOPP}(s^i)|} = \frac{1}{p \sum_{s \in S} \pi(s) |\mathrm{OPP}(s)|}. \tag{3.2}$$

Since the long-run average cycle time for a packet, to be denoted by TIME(n, k, p), can be defined to be equal to the long-run average time for nk units progress of the system, we have

$$\mathrm{TIME}(n, k, p) = \frac{nk}{pE(|\mathrm{OPP}|)}, \tag{3.3}$$

where $E(|\mathrm{OPP}|) = \sum_{s \in S} |\mathrm{OPP}(s)| \pi(s)$, the expected number of opportunities under π.

To see how the strong law is used to calculate the limit in (3.1), it helps to consider the associated Markov chain, with state space $S \times S$, which generates the sequence of states

$$(s^0, s^1), (s^1, s^2), \ldots, (s^{t-1}, s^t), \ldots.$$

It is easily verified that this Markov chain has stationary distribution π^* defined by

$$\pi^*(s, s') = \pi(s)P(s, s'),$$

where π and P are defined by (2.7) and (2.1), respectively. By an application of the strong law (Chung [2, Theorem 2, §I.15]), the limit (3.1) can now be seen to equal, with probability one,

$$\frac{1}{\sum_{(s,s')\in S\times S} |\mathrm{OPP}(s) \cap \mathrm{POSTOPP}(s')|\pi^*(s, s')}.$$

This is the same as

$$\begin{aligned}
&\frac{1}{\sum_{s\in S}\sum_{A\subseteq \mathrm{OPP}(s)} |A|\pi(s)P(s, s^A)} \\
&= \frac{1}{\sum_{s\in S}\pi(s)\sum_{\ell=1}^{|\mathrm{OPP}(s)|}\ell\binom{|\mathrm{OPP}(s)|}{\ell}p^\ell(1-p)^{|\mathrm{OPP}(s)|-\ell}} \\
&= \frac{1}{\sum_{s\in S}\pi(s)p|\mathrm{OPP}(s)|} \\
&= \frac{1}{p\cdot\sum_{s\in S}\pi(s)|\mathrm{OPP}(s)|},
\end{aligned}$$

where the second-to-last equality holds because the inner sum is a binomial expectation. The verification of (3.2) is now complete.

The following proposition calculates $E(|\mathrm{OPP}|)$ explicitly in terms of n, k, and p.

PROPOSITION 3.1.

$$E(|\mathrm{OPP}|) = \frac{\sum_{\ell=1}^{\min(k,n-k)}\binom{k-1}{\ell-1}\binom{n-k-1}{\ell-1}(1-p)^{-\ell}}{\sum_{\ell=1}^{\min(k,n-k)}\frac{1}{\ell}\binom{k-1}{\ell-1}\binom{n-k-1}{\ell-1}(1-p)^{-\ell}}. \tag{3.4}$$

Proof. Since $k < n$, for every $s \in S$ there must exist j such that $s_js_{j+1} = 10$. Hence $|\mathrm{OPP}(s)| \geq 1$. Moreover,

$$\mathrm{OPP}(s) \subseteq \{j : 0 \leq j \leq n-1 \text{ and } s_j = 1\},$$

so $|\mathrm{OPP}(s)| \leq k$. Further,

$$\{j+1 : j \in \mathrm{OPP}(s)\} \subseteq \{i : 1 \leq i \leq n \text{ and } s_i = 0\},$$

so $|\mathrm{OPP}(s)| \leq n-k$. We have thus shown that

$$1 \leq |\mathrm{OPP}(s)| \leq \min(k, n-k)$$

for each $s \in S$. For each integer ℓ such that $1 \leq \ell \leq \min(k, n-k)$, let $S_\ell = \{s \in S : |\mathrm{OPP}(s)| = \ell\}$. By Corollary 2.4, we have

$$\begin{aligned}
E(|\mathrm{OPP}|) &= \frac{\sum_{s\in S}|\mathrm{OPP}(s)|\gamma(s)}{\sum_{s\in S}\gamma(s)} \\
&= \frac{\sum_{\ell=1}^{\min(k,n-k)}\ell|S_\ell|(1-p)^{-\ell}}{\sum_{\ell=1}^{\min(k,n-k)}|S_\ell|(1-p)^{-\ell}}.
\end{aligned}$$

The proof will be complete if we can show that

$$|S_\ell| = \frac{n}{\ell}\binom{k-1}{\ell-1}\binom{n-k-1}{\ell-1} \quad \text{for } 1 \le \ell \le \min(k, n-k). \tag{3.5}$$

Note that every $s \in S$ is the concatenation of an alternating sequence of nonempty blocks of 0's and nonempty blocks of 1's, so $s \in S_\ell$ if and only if it has one of the following four possible forms:

(i) s is the concatenation of an alternating sequence of ℓ nonempty blocks of 1's and ℓ nonempty blocks of 0's (where s starts with a block of 1's);

(ii) s is the concatenation of an alternating sequence of $\ell+1$ nonempty blocks of 1's and ℓ nonempty blocks of 0's (where s starts with a block of 1's);

(iii) s is the concatenation of an alternating sequence of ℓ nonempty blocks of 0's and ℓ nonempty blocks of 1's (where s starts with a block of 0's);

(iv) s is the concatenation of an alternating sequence of $\ell+1$ nonempty blocks of 0's and ℓ nonempty blocks of 1's (where s starts with a block of 0's).

Since any string of length a can be partitioned into b nonempty blocks in $\binom{a-1}{b-1}$ ways and since each $s \in S$ has k 1's and $(n-k)$ 0's, we have

$$\begin{aligned} |S_\ell| &= \binom{k-1}{\ell-1}\binom{n-k-1}{\ell-1} + \binom{k-1}{\ell}\binom{n-k-1}{\ell-1} \\ &\quad + \binom{k-1}{\ell-1}\binom{n-k-1}{\ell-1} + \binom{k-1}{\ell-1}\binom{n-k-1}{\ell} \\ &= \left(1 + \frac{k-\ell}{\ell} + 1 + \frac{n-k-\ell}{\ell}\right)\binom{k-1}{\ell-1}\binom{n-k-1}{\ell-1} \\ &= \frac{n}{\ell}\binom{k-1}{\ell-1}\binom{n-k-1}{\ell-1}. \end{aligned}$$

(Similar counting arguments are discussed in detail in Johnson and Kotz [4], for example.) □

THEOREM 3.2.

$$\text{TIME}(n, k, p) = \frac{nk}{p} \frac{\sum_{\ell=1}^{\min(k,n-k)} \frac{1}{\ell} \cdot \binom{k-1}{\ell-1}\binom{n-k-1}{\ell-1}(1-p)^{-\ell}}{\sum_{\ell'=1}^{\min(k,n-k)} \binom{k-1}{\ell'-1}\binom{n-k-1}{\ell'-1}(1-p)^{-\ell'}}. \tag{3.6}$$

Proof. This is an immediate consequence of (3.3) and Proposition 3.1. □

4. Optimum throughput. Given a station j, if $P(\text{TRANS at } j)$ denotes the probability, under the stationary distribution π, that a packet gets transmitted to station $(j+1)$, then

$$P(\text{TRANS at } j) = p \cdot \pi\{s \in S : j \in \text{OPP}(s)\}.$$

Since $\pi\{s \in S : j \in \text{OPP}(s)\}$ is independent of j, so is $P(\text{TRANS at } j)$, and we can denote it simply by $P(\text{TRANS})$. It now follows that

$$n \cdot P(\text{TRANS}) = p \sum_{j=0}^{n-1} \pi\{s \in S : j \in \text{OPP}(s)\}.$$

Therefore,

$$n \cdot P(\text{TRANS}) = p \cdot E(|\text{OPP}|). \tag{4.1}$$

The last equality is seen by expressing |OPP| as a sum of n 0–1 valued random variables. Either of the equivalent expressions in (4.1) can be taken as the definition of the "throughput" of the system.

We now solve the following optimization problem. For fixed n and p we determine the value of k that maximizes P(TRANS). Hence, we determine the value of k that maximizes throughput.

THEOREM 4.1. *Let n, a positive integer, and p, a real number such that $0 < p < 1$, be fixed. Then as a function of k, P(TRANS) is maximized at k_0, where*

$$k_0 = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{n-1}{2} \text{ or } \frac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases} \tag{4.2}$$

The proof of Theorem 4.1 relies on a lemma.

LEMMA 4.2. *Let μ_1 and μ_2 be functions defined on $\{1, \dots, L\}$, where L is a positive integer such that*

(i) $\mu_i(\ell) \geq 0$ *for all ℓ and for $i = 1, 2$;*

(ii) $\mu_i(\ell) = 0$ *implies* $\mu_i(\ell') = 0$ *for all $\ell' \geq \ell$ and for $i = 1, 2$;*

(iii) $\sum_{\ell=1}^{L} \mu_i(\ell) = 1$, *for $i = 1, 2$; and*

(iv) $\mu_1(\ell+1)/\mu_1(\ell) \geq \mu_2(\ell+1)/\mu_2(\ell)$ *for all ℓ such that $\mu_1(\ell) > 0$ and $\mu_2(\ell) > 0$.*

Let φ be a nondecreasing function on $\{1, \dots, L\}$. Then

$$\sum_{\ell=1}^{L} \varphi(\ell)\mu_1(\ell) \geq \sum_{\ell=1}^{L} \varphi(\ell)\mu_2(\ell).$$

[*Remark.* This elementary lemma is a special case of a well-known result in the monotone likelihood ratio theory in statistics (see Lehmann [5, §3.3]). However, for completeness, we include a proof.]

Proof. Let ℓ_0 be the least integer in $\{1, \dots, L\}$ such that $\mu_1(\ell_0) \geq \mu_2(\ell_0)$. Such an integer must exist because of hypothesis (iii). Now hypotheses (iv), (ii), and (i) imply that

$$\mu_1(\ell) \geq \mu_2(\ell) \quad \text{for all } \ell \geq \ell_0. \tag{4.3}$$

If $\ell_0 = 1$, then by (iii) and (4.3), $\mu_1(\ell) = \mu_2(\ell)$ for all ℓ and the assertion of the lemma is true. If $\ell_0 \geq 2$,

$$\begin{aligned}
&\sum_{\ell=1}^{L} \varphi(\ell)[\mu_1(\ell) - \mu_2(\ell)] \\
&\quad = \sum_{\ell=1}^{\ell_0-1} \varphi(\ell)[\mu_1(\ell) - \mu_2(\ell)] + \sum_{\ell=\ell_0}^{L} \varphi(\ell)[\mu_1(\ell) - \mu_2(\ell)] \\
&\quad \geq \varphi(\ell_0 - 1) \cdot \sum_{\ell=1}^{\ell_0-1} [\mu_1(\ell) - \mu_2(\ell)] + \varphi(\ell_0) \sum_{\ell=\ell_0}^{L} [\mu_1(\ell) - \mu_2(\ell)] \\
&\qquad\qquad [\text{since } \varphi \text{ is nondecreasing}] \\
&\quad = [\varphi(\ell_0) - \varphi(\ell_0 - 1)] \sum_{\ell=\ell_0}^{L} [\mu_1(\ell) - \mu_2(\ell)] \quad [\text{by (iii)}] \\
&\quad \geq 0 \quad [\text{by (4.3)}]. \qquad \square
\end{aligned}$$

Proof of Theorem 4.1. Because of (4.1) and (3.4), maximizing $P(\text{TRANS})$ is equivalent to minimizing

$$\sum_{\ell=1}^{\min(k,n-k)} \frac{1}{\ell} \cdot \frac{\binom{k-1}{\ell-1}\binom{n-k-1}{\ell-1}(1-p)^{-\ell}}{\sum_{\ell'=1}^{\min(k,n-k)} \binom{k-1}{\ell'-1}\binom{n-k-1}{\ell'-1}(1-p)^{-\ell'}}.$$

For each positive integer $k < n$, define

$$\mu_k(\ell) = \frac{\binom{k-1}{\ell-1}\binom{n-k-1}{\ell-1}(1-p)^{-\ell}}{\sum_{\ell'=1}^{\min(k,n-k)} \binom{k-1}{\ell'-1}\binom{n-k-1}{\ell'-1}(1-p)^{-\ell'}} \tag{4.4}$$

if $1 \le \ell \le \min(k, n-k)$ and define $\mu_k(\ell) = 0$ if $\min(k, n-k) < \ell \le \min(k_0, n-k_0)$.

Since k_0 satisfies (4.2), it follows that for $k < n$ the functions μ_{k_0} and μ_k satisfy hypotheses (i), (ii), and (iii) of Lemma 4.2. Also,

$$\frac{\mu_k(\ell+1)}{\mu_k(\ell)} = \frac{\binom{k-1}{\ell}\binom{n-k-1}{\ell}(1-p)^{-\ell-1}}{\binom{k-1}{\ell-1}\binom{n-k-1}{\ell-1}(1-p)^{-\ell}} = \frac{k(n-k) - \ell n + \ell^2}{\ell^2} \cdot \frac{1}{1-p}$$

for $1 \le \ell < \min(k, n-k)$. Therefore,

$$\frac{\mu_{k_0}(\ell+1)}{\mu_{k_0}(\ell)} \ge \frac{\mu_k(\ell+1)}{\mu_k(\ell)}$$

for all $k < n$ and for all ℓ satisfying $1 \le \ell \le \min(k, n-k)$. Thus hypothesis (iv) of Lemma 4.2 is also satisfied. So by applying Lemma 4.2 to the nondecreasing function φ defined by $\varphi(\ell) = -\frac{1}{\ell}$, we obtain

$$\sum_{\ell=1}^{\min(k_0,n-k_0)} \frac{1}{\ell}\mu_{k_0}(\ell) \le \sum_{\ell=1}^{\min(k,n-k)} \frac{1}{\ell}\mu_k(\ell) \quad \text{for all } k < n.$$

This completes the proof of the theorem. □

5. Concluding remarks.

5.1. A bound for TIME(2N,N,1/2). In Berman and Simon [1], the question of obtaining an exact expression for TIME$(2N, N, 1/2)$ was raised. Of course, Theorem 3.2 gives the answer. Although the expression (3.6) in Theorem 3.2 is complicated, here is an easy argument to show that

$$\text{TIME}\left(2N, N, \frac{1}{2}\right) \le \frac{4}{p}N.$$

If μ_p denotes the function defined by the right side of (4.4) when $k = N$ and $n = 2N$ and μ_0 denotes this function when, in addition, $p = 0$, then Lemma 4.2 can be applied routinely to show that

$$\sum_{\ell=1}^{N} \frac{1}{\ell}\mu_p(\ell) \le \sum_{\ell=1}^{N} \frac{1}{\ell}\mu_0(\ell).$$

Hence

$$\begin{aligned}
\mathrm{TIME}(2N, N, p) &= \frac{2N^2}{p}\sum_{\ell=1}^{N}\frac{1}{\ell}\mu_p(\ell)\\
&\le \frac{2N^2}{p}\sum_{\ell=1}^{N}\frac{1}{\ell}\mu_0(\ell)\\
&= \frac{2N}{p}\frac{\sum_{\ell=1}^{N}\frac{N}{\ell}\cdot\binom{N-1}{\ell-1}\binom{N-1}{\ell-1}}{\sum_{\ell'=1}^{N}\binom{N-1}{N-\ell'}\binom{N-1}{\ell'-1}}\\
&= \frac{2N}{p}\frac{\sum_{\ell=1}^{N}\binom{N}{N-\ell}\binom{N-1}{\ell-1}}{\sum_{\ell'=1}^{N}\binom{N-1}{N-\ell'}\binom{N-1}{\ell'-1}}\\
&= \frac{2N}{p}\cdot\frac{\binom{2N-1}{N-1}}{\binom{2N-2}{N-1}}\\
&\le \frac{4}{p}N.
\end{aligned}$$

5.2. Asymptotics. The expression for TIME(n, k, p) given by (3.6) is hard to evaluate if k and $n-k$ are large. We tried approximating it by the technique of moment-generating functions as follows:

Rewrite (3.6) as

$$\mathrm{TIME}(n, k, p) = \frac{nk}{p}\cdot\frac{\sum_{\ell=1}^{\min(k,n-k)}\binom{k}{k-\ell}\binom{n-k-1}{\ell-1}(1-p)^{-\ell}}{\sum_{\ell'=1}^{\min(k,n-k)}\ell'\binom{k}{k-\ell'}\binom{n-k-1}{\ell'-1}(1-p)^{-\ell'}}. \tag{5.1}$$

Consider now an urn with k red balls and $(n-k-1)$ black balls. Let X denote the number of black balls drawn when $(k-1)$ balls are drawn from the urn, one by one, without replacement. If $t=(1-p)^{-1}$, we may now write (5.1) as

$$\begin{aligned}
\mathrm{TIME}(n, k, p) &= \frac{nk}{p}\frac{E(t^{X+1})}{E((X+1)t^{X+1})} \quad \text{[where } E \text{ denotes expectation]}\\
&= \frac{nk}{p}\frac{E(t^{X+1})}{t\cdot\frac{d}{dt}[E(t^{X+1})]}.
\end{aligned}$$

This last expression can be approximated by regarding the hypergeometric random variable X to be approximately normally distributed. In particular, this technique yields for TIME$(2N, N, p)$ the approximation

$$\frac{N}{p}\cdot\frac{1}{\frac{1}{2N}+\frac{1}{4}-\frac{1}{16}\ln(1-p)}.$$

While this approximation seems to work reasonably well for $p \le 0.5$ as compared to simulated values, it is very poor for values of p close to 1.

It would be desirable to derive a closed-form expression for the limit of

$$\frac{n}{\mathrm{TIME}(n, k, p)}$$

as n approaches ∞, k approaches ∞, and $\frac{k}{n}$ approaches α where $0 < \alpha < 1$. In the particular case where $\alpha = 1/2$, Berman and a referee have conjectured that this limit equals $1 - \sqrt{1-p}$.

Acknowledgments. Shortly after we obtained our results, L. A. Shepp obtained (3.6) independently. A referee provided a number of detailed comments that led to a substantial change in style as well as shortening of an earlier version.

REFERENCES

[1] P. BERMAN AND J. SIMON, *Investigations of fault-tolerant networks of computers* (preliminary version), in Proceedings 20th ACM Symposium on Theory of Computing, Chicago, ACM Press, May 1988, pp. 66–77.

[2] K. L. CHUNG, *Markov Chains with Stationary Transition Probabilities*, second ed., Springer-Verlag, Berlin 1967.

[3] W. FELLER, *An Introduction to Probability Theory and its Applications*, Vol. I, Third ed., Wiley & Sons, New York, 1968.

[4] N. JOHNSON AND S. KOTZ, *Urn Models and Their Application*, Wiley and Sons, New York, 1977.

[5] E. LEHMANN, *Testing Statistical Hypotheses*, Wiley and Sons, New York, 1959.

[6] A. TANENBAUM, *Computer Networks*, Prentice-Hall, Englewood Cliffs, NJ, 1981.