More NP-Complete Problems

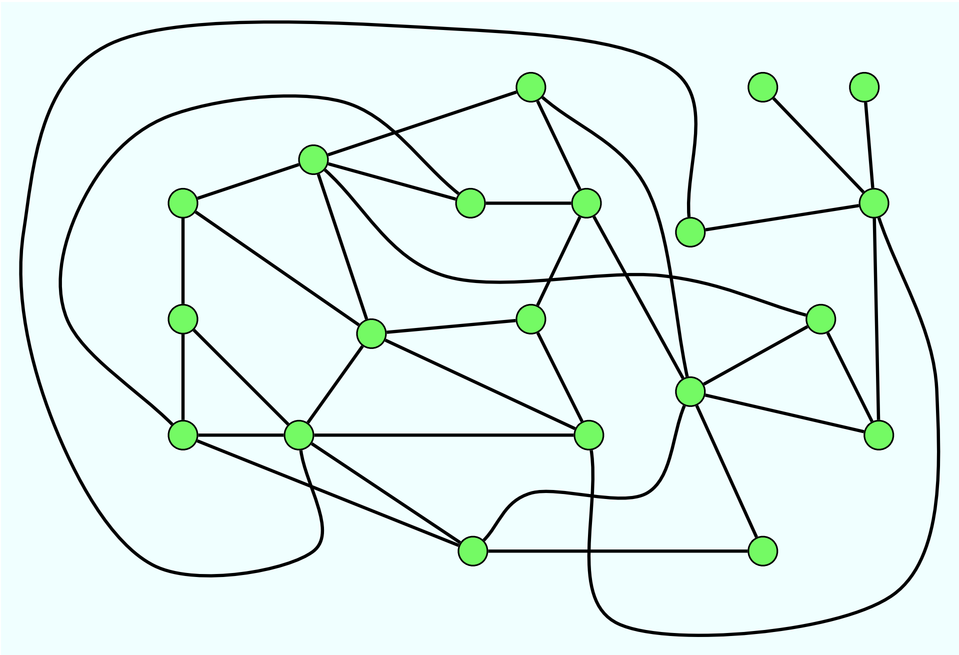
Clique

We know: $\mathit{3SAT} \leq_P \mathit{CLIQUE}$, $\mathit{CLIQUE} \in \mathbf{NP}$, and $\mathit{3SAT}$ is $\mathbf{NP}$-complete. So, CLIQUE is $\mathbf{NP}$-complete.

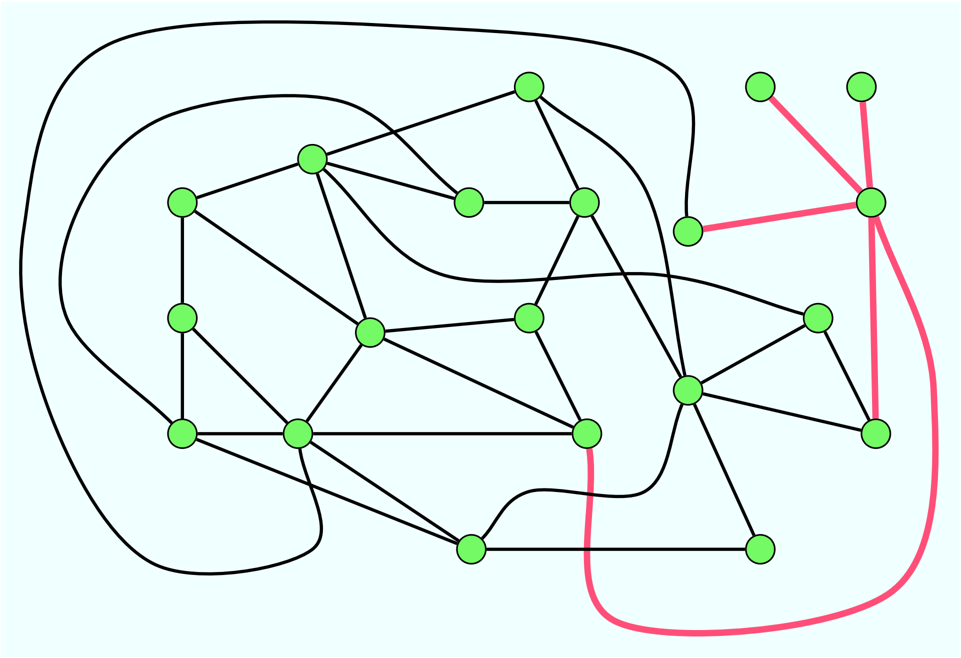
Vertex Cover

A vertex cover of an undirected graph is a subset of nodes such that every edge touches a member of the subset.

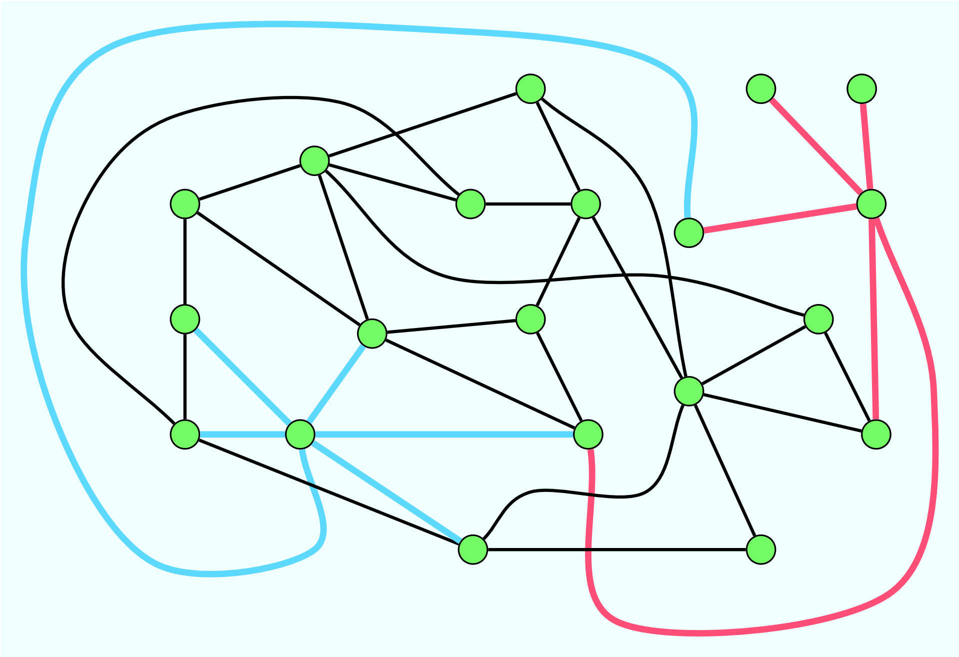
Example: a 10-node Vertex Cover



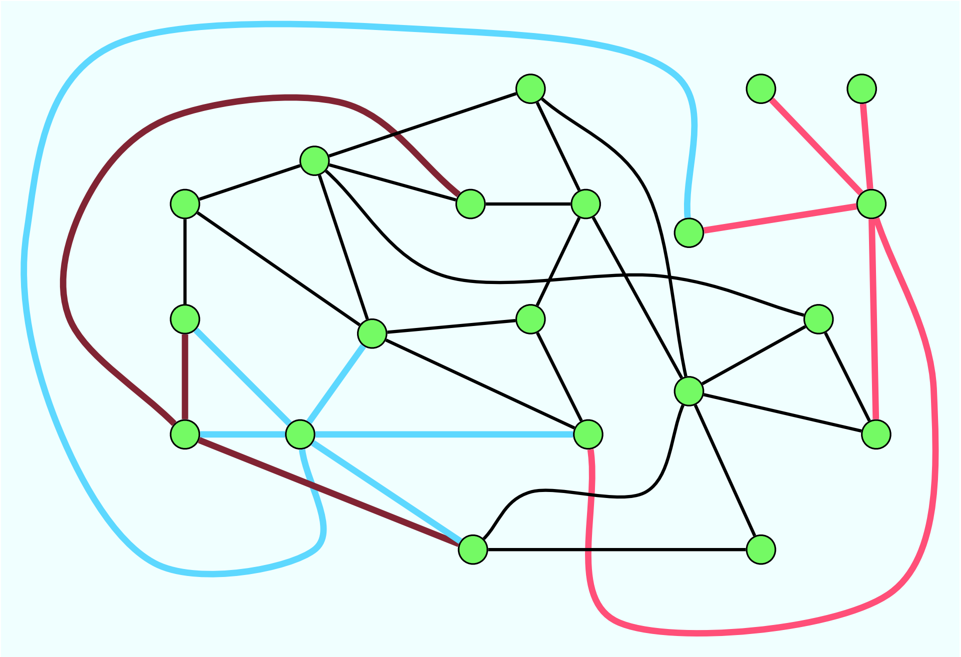
Example: Step 1



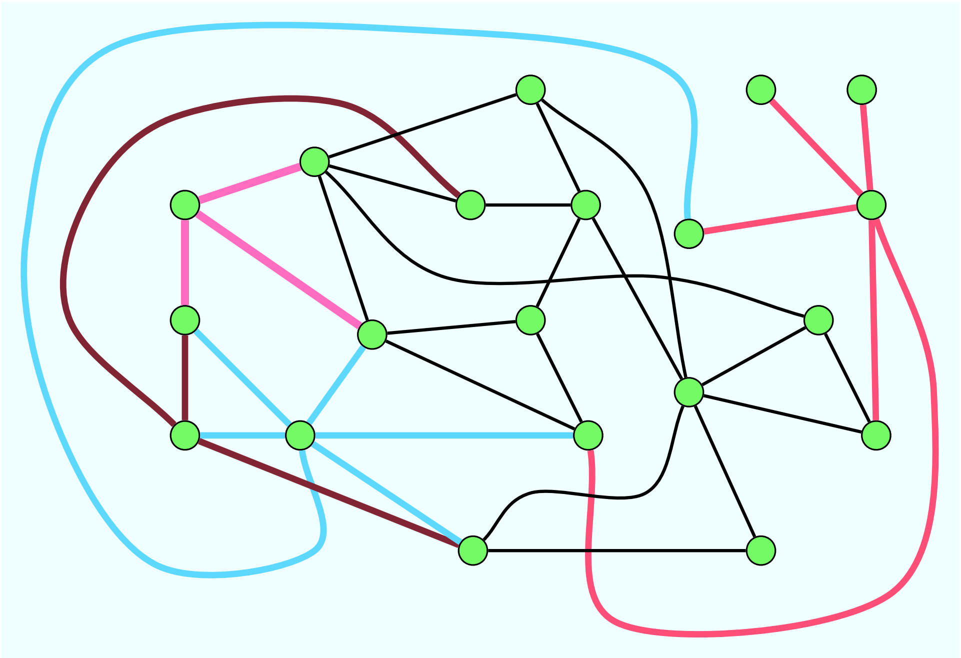
Example: Step 2



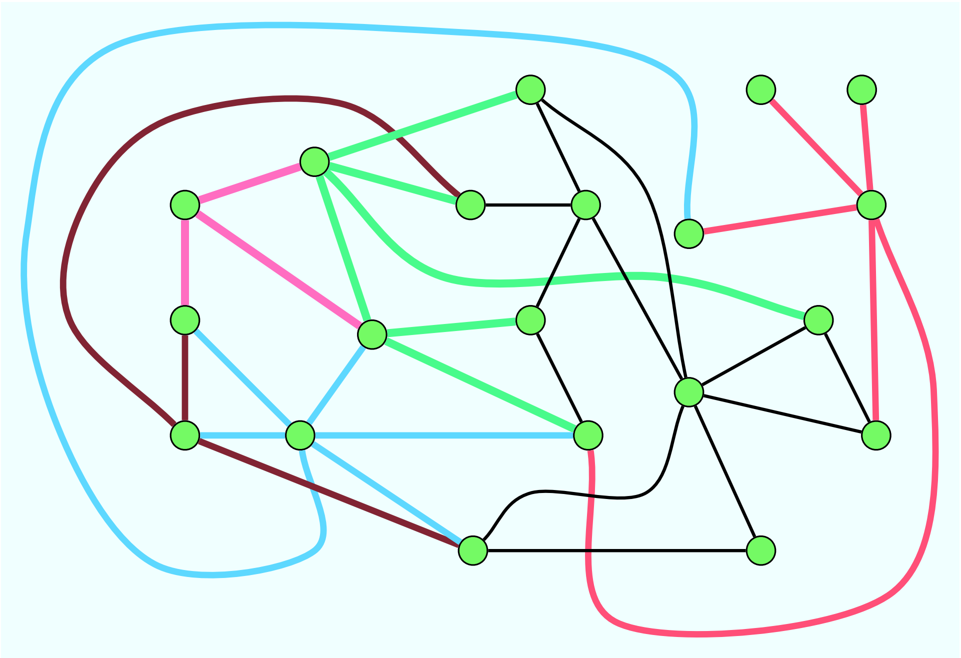
Example: Step 3



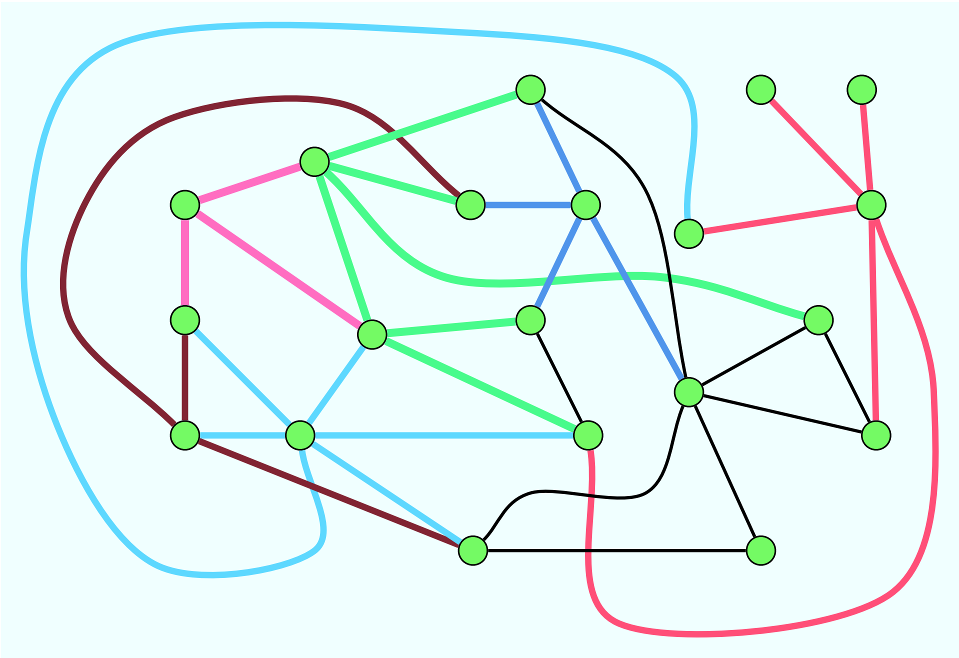
Example: Step 4



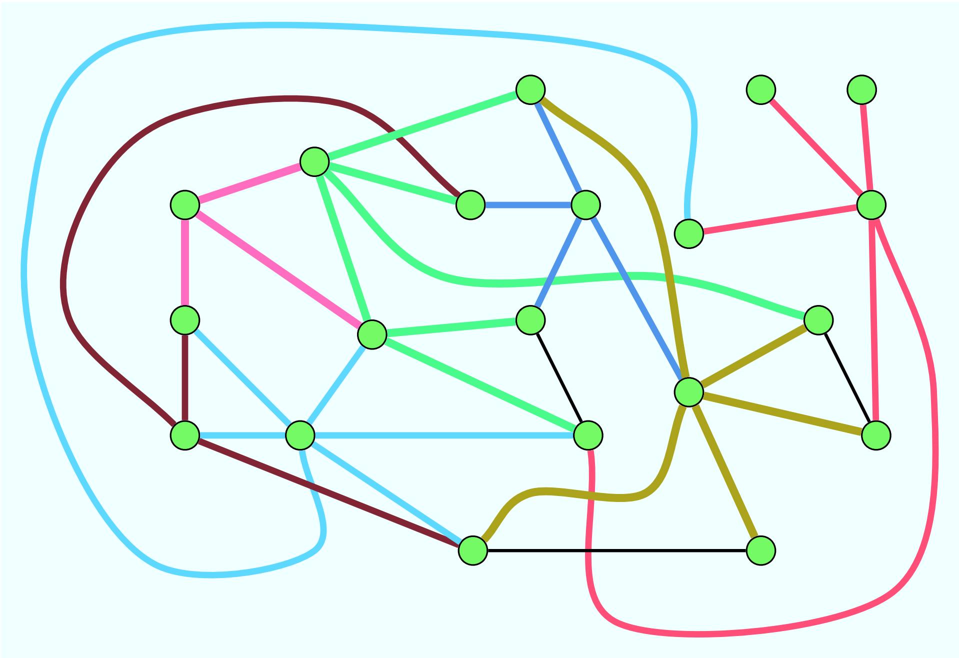
Example: Step 5



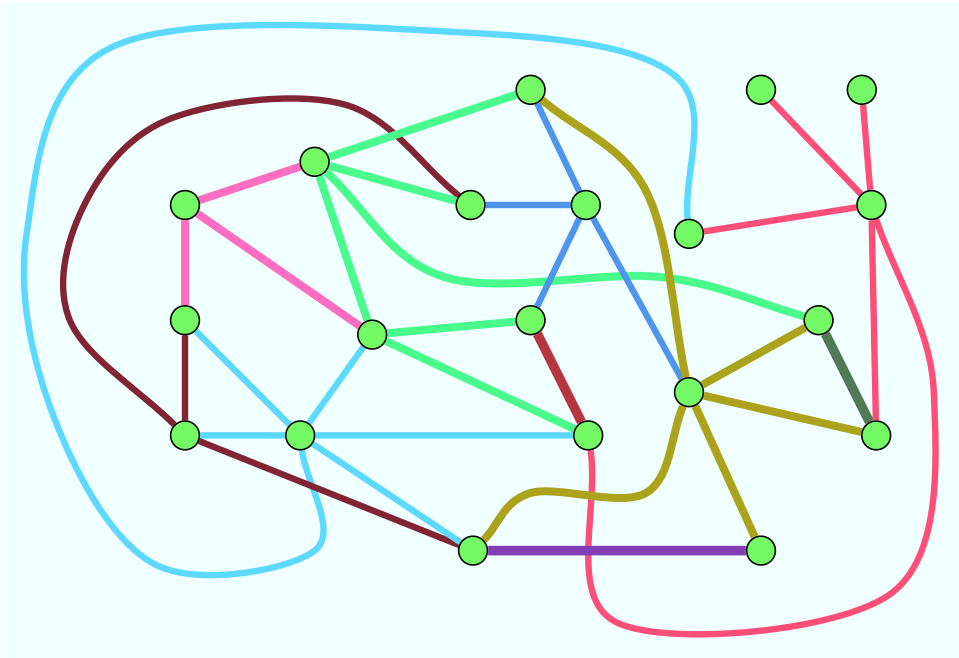
Example: Step 6



Example: Step 7



Example: Steps 8, 9, and 10



Vertex Cover Is NP-Complete

$VERTEX\text{-}COVER = \{\langle G, k\rangle \mid G$ has a vertex cover of size $k\}$.

Theorem. *$VERTEX\text{-}COVER$* **is NP-complete.**

Proof

Proving $VERTEX\text{-}COVER \in \mathbf{NP}$ is easy. Guess a bit for each node to decide whether or not to select the node. Then check whether exactly k nodes have been selected, if so, check whether the k nodes selected form a vertex cover.

Proof

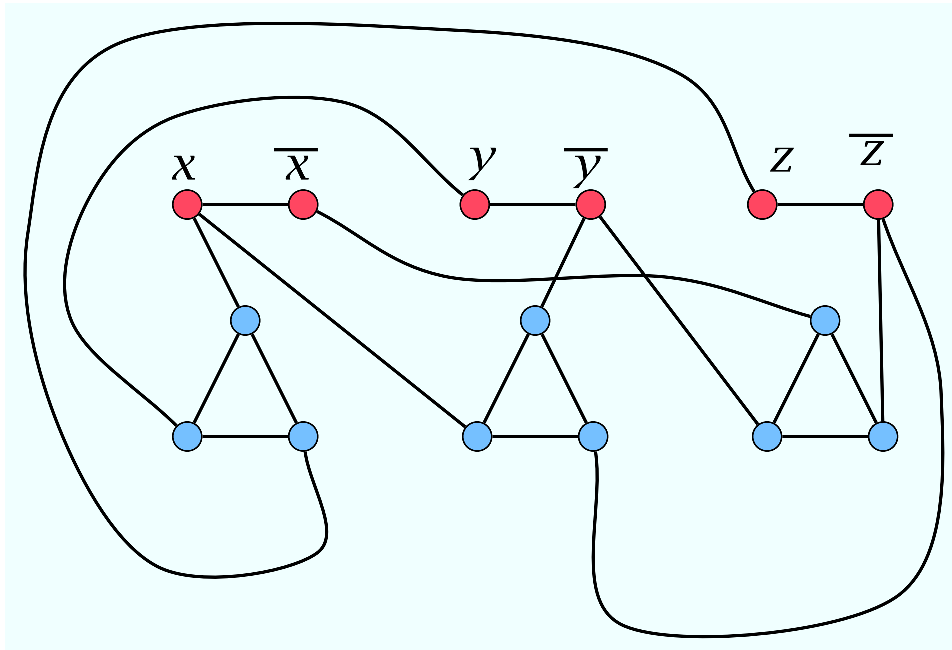
Reduce *3SAT* to *VERTEX-COVER*.

Let ϕ be an instance of *3SAT* with n variables and m clauses. Define the graph G as follows:

- **The Nodes**
  - **the assignments:** $v_i, \overline{v_i}, 1 \leq i \leq n$;
  - **the literals:** $a_{i1}, a_{i2}, a_{i3} : 1 \leq i \leq m$
- **The Edges**
  - assignment pairs: $(v_i, \overline{v_i})$, $1 \leq i \leq n$;
  - literal triangles: $(a_{i1}, a_{i2}), (a_{i2}, a_{i3}), (a_{i3}, a_{i1})$, $1 \leq i \leq m$;
  - literal-assignment pairs: for each i, $1 \leq i \leq n$, and j, $1 \leq j \leq 3$, connect a_{ij} and its corresponding assignment.

Example

The graph for $(x \vee y \vee z) \wedge (x \vee \overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{y} \vee \overline{z})$.



Proof (cont'd)

We claim that G has an $(n + 2m)$ node vertex cover if and only if ϕ is in $\mathit{3SAT}$.

- There are exactly n edges of the type $(v_i, \overline{v_i})$, $1 \leq i \leq n$. So a cover has to have at least one node out of v_i and $\overline{v_i}$ for every i.

Proof (cont'd)

We claim that G has an $(n + 2m)$ node vertex cover if and only if ϕ is in $\mathit{3SAT}$.

- From each assignment pair, $(v_i, \overline{v_i})$, at least one node has to be chosen.
- From each literal triangle $\triangle a_{i1} a_{i2} a_{i3}$, at least two nodes have to be chosen.

The total required number of nodes to be selected is $n + 2m$.

Thus, any $n+2m$-node vertex cover must select 2 nodes per literal triangle and 1 node per assignment pair.

Proof (cont'd)

If exactly 2 nodes are selected from a triangle, then all the edges attached to the triangle nodes are covered except for one, which is one that connects between:

- the triangle node that is NOT chosen and
- the literal node corresponding to that unchosen node.

To cover that edge, the corresponding literal node has to be chosen.

Proof (cont'd)

If a selection of $n+2m$ nodes is a cover then for each triangle there is at least one node whose other end point is selected. Since we are select exactly one of x and $\overline{x}$ for each literal pair, it means that the selections on the literal pairs is a satisfying assignment. that is connected to

- for each triangle, the other end of the literal-assignment pair incident at the node that is not selected is selected.

  For example, if a_{i1} is not selected and is connected to v_r, then v_r has to be selected.

So an $n+2m$-node vertex cover exists if and only if the selected assignment nodes form a satisfying assignment of the formula. ∎

Subset-Sum is NP-complete

$SUBSET\text{-}SUM$ is the problem of, given a multiset of numbers $z_1, \ldots, z_m$ and a number S, whether there is subset $y_1, \ldots, y_t$ of z_i's such that $y_1 + \cdots + y_t = S$.

Subset-Sum is NP-complete

$SUBSET\text{-}SUM$ is the problem of, given a multiset of numbers $z_1, \ldots, z_m$ and a number S, whether there is subset $y_1, \ldots, y_t$ of z_i's such that $y_1 + \cdots + y_t = S$.

Theorem. ***$SUBSET\text{-}SUM$ is NP-complete.***

Subset-Sum is NP-complete

$\mathit{SUBSET\text{-}SUM}$ is the problem of, given a multiset of numbers $z_1, \ldots, z_m$ and a number S, whether there is subset $y_1, \ldots, y_t$ of z_i's such that $y_1 + \cdots + y_t = S$.

Theorem. ***$\mathit{SUBSET\text{-}SUM}$* is NP-complete.**

Proof Reduce $\mathit{3SAT}$ to $\mathit{SUBSET\text{-}SUM}$. The construction is reminiscent of the reduction from $\mathit{3SAT}$ to $\mathit{VERTEX\text{-}COVER}$, where the reduction generates a graph whose $n + 2m$ node cover has a property that at least one "literal-occurrence" edge of each triangle is touched and the rest of the nodes in each triangle is touched.

Proof (cont'd)

Let ϕ be a formula of n variables and m clauses. Introduce decimal numbers $y_1, \ldots, y_n$, $z_1, \ldots, z_n$, $c_1, \ldots, c_m$, $d_1, \ldots, d_m$, each of at most $n + m$ digits.

y_i y_i has a 1 at the $(m+1)$st digit and has a 1 at position j if x_i appears in the jth clause; all the other positions have a 0

z_i z_i has a 1 at the $(m+1)$st digit and has a 1 at position j if $\overline{x_i}$ appears in the jth clause; all the other positions have a 0

c_i, d_i c_i has a 1 only at the ith position, d_i has a 1 only at the ith position,

S S is the number that has a 3 at every position between 1 and m and has a 1 at every position between $m+1$ and $m+n$

Example: $(x \vee y \vee z) \wedge (x \vee \overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{y} \vee \overline{z})$

Clauses									variables					
	1	1		1	1		1	1		1		1		1
					1			1						1
		1												1
								1				1		
		1			1							1		
								1		1				
		1			1					1				
								1						
								1						
					1									
					1									
		1												
		1												

Proof (cont'd)

In order to generate S, exactly one of y_i and z_i has to be selected for every i so that the selection as a whole touches each bit position between 1 and m at least once (and at most three times). Such a selection is a satisfying assignment of ϕ. ∎