Variants of Turing Machines

Multitape TMs

A **multitape Turing machine** is a Turing machine with additional tapes with each tape is accessible individually, with the input on the first tape, and with the others blank at the beginning.

For a k-tape Turing machine, the transition δ is a mapping from $Q \times \Gamma^k$ to $Q \times \Gamma^k \times \{L, R\}^k$.

Nondeterministic TMs

A **nondeterministic Turing machine** is one in which the transition is mapping to the power set of $Q \times \Gamma \times \{L, R\}$.

A nondeterministic Turing machine **accepts** an input if it enters an accepting state **for some computation path.**

Equivalence Between Single-tape TMs and Multitape TMs

Theorem. **Every multitape Turing machine has an equivalent single-tape Turing machine.**

Equivalence Between Single-tape TMs and Multitape TMs

Theorem. **Every multitape Turing machine has an equivalent single-tape Turing machine.**

Proof From a k-tape TM M build a single-tape simulator S.

Theorem. **Every multitape Turing machine has an equivalent single-tape Turing machine.**

Proof From a k-tape TM M build a single-tape simulator S.

The main idea is to use the tape available to represent

- the contents of the tape squares that the head has ever visited for each tape, including the entire squares that initially hold the input,
- and the current head position for each tape.

Create such a representation for each tape and connect them with a delimiter in between, at the beginning, and at the end.

Tape Encoding

For each $a \in \Gamma$, let $\widetilde{a}$ be a new symbol to signify that **a head is located on the symbol.**

- The input tape $w_1 \cdots w_n \sqcup \cdots$ with the head scanning the first symbol (this occurs at the beginning)

$$\widetilde{w_1} w_2 \cdots w_n.$$

Tape Encoding

- If a tape holds $a_1 \cdots a_s \sqcup \cdots$ and the farthest position the head has traveled is $t > r$.
  - If the head position is $r < s$, then its representation is:

$$a_1 \cdots a_{r-1} \widetilde{r_s} a_r \cdots a_s \underbrace{\sqcup \cdots \sqcup}_{t-s}.$$

  - If the head position is $r > s$, then its representation is:

$$a_1 \cdots a_s \underbrace{\sqcup \cdots \sqcup}_{r-s-1} \widetilde{\sqcup} \underbrace{\sqcup \cdots \sqcup}_{t-r}.$$

The encoding never decreases in length.

Delimiter

Use a new symbol $\#$ as a **delimiter.**

On input $w = w_1 \cdots w_n$, the initial form of encoding is:

$$\#\widetilde{w_1}w_2 \cdots w_n\#\widetilde{\sqcup}\#\widetilde{\sqcup}\# \ \cdots \ \#\widetilde{\sqcup}\#$$

S's Action

Memorize M's state using a state.

1. Construct the initial form.
2. Repeat the following:
   (a) If M has accepted or rejected, accept or reject accordingly.
   (b) Otherwise, scan the tape in a direction and record, using the state, **the symbols being scanned by the heads of M**.
   (c) **Determine the next move** of M.
   (d) **Modify the encoding accordingly.** Insert symbols if necessary.
   (e) **Change the state accordingly.**

Identification of Symbols Scanned

In the case when the tape is scanned from right to left, use the following states.

1. $(p_{\text{scan}}, q, a_1, \ldots, a_k), a_1, \ldots, a_k \in \Gamma$: This means that the current state is q and the symbols being scanned are $a_1, \ldots, a_k$.
2. $(p_{\text{scan}}, q, ?, \ldots, ?, a_{r+1}, \ldots, a_k), a_{r+1}, \ldots, a_k \in \Gamma$: This means that for the first r tapes the symbols being scanned are yet to be identified but for the others the symbols have been identified to be $a_{r+1}, \ldots, a_k$.

Identification of Symbols Scanned

In the case when the tape is scanned from right to left, use the following states.

1. $(p_{\rm scan}, q, a_1, \ldots, a_k), a_1, \ldots, a_k \in \Gamma$: This means that the current state is q and the symbols being scanned are $(p_{\rm scan}, q, ?, \ldots, ?, a_{r+1}, \ldots, a_k), a_{r+1}, \ldots, a_k \in \Gamma$: This means that for the first r tapes the symbols being scanned are yet to be identified but for the others the symbols have been identified to be $a_{r+1}, \ldots, a_k$.

Start scanning from the end in state $(p_{\rm scan}, q, ?, \ldots, ?)$.

Each time a symbol of the form $\widehat{X}$ is encountered, replace the rightmost $?$ with that X.

When all $?$'s are gone, S knows the action of M.

If forward motion is used, the symbols are from left to right.

Tape Modification

This step consists of

- rewriting the symbol on the current head position and
- rewriting the symbols around the current head position to move the head to the left or to the right.

Tape Modification Rules

- If the current contents are $\cdots a\widetilde{b}\cdots$, $a \neq \#$, b is to be replaced by b', and the head moves to the left, then replace the two symbols by $\widetilde{a}b'$.
- If the current contents are $\cdots \#\widetilde{b}\cdots$, b is to be replaced by b', and the head moves to the left, then replace the two symbols by $\#\widetilde{b'}$.
- If the current contents are $\cdots \widetilde{b}a\cdots$, $a \neq \#$, b is to be replaced by b', and the head moves to the right, then replace the two symbols by $b'\widetilde{a}$.

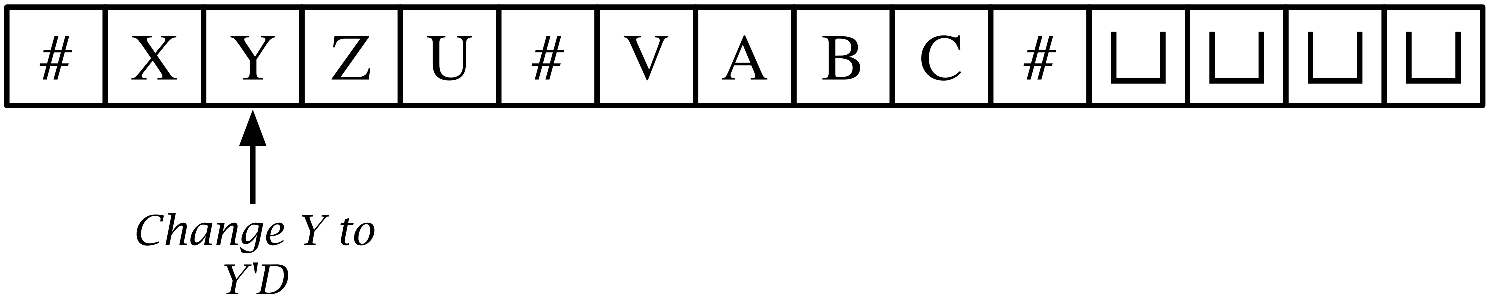
Tape Modification Rules (cont'd)

- If the current contents are $\cdots \widetilde{b}\# \cdots$, b is to be replaced by b', and the head moves to the right, then replace the $\widetilde{b}\#$ by $b'\widetilde{\sqcup}\#$.

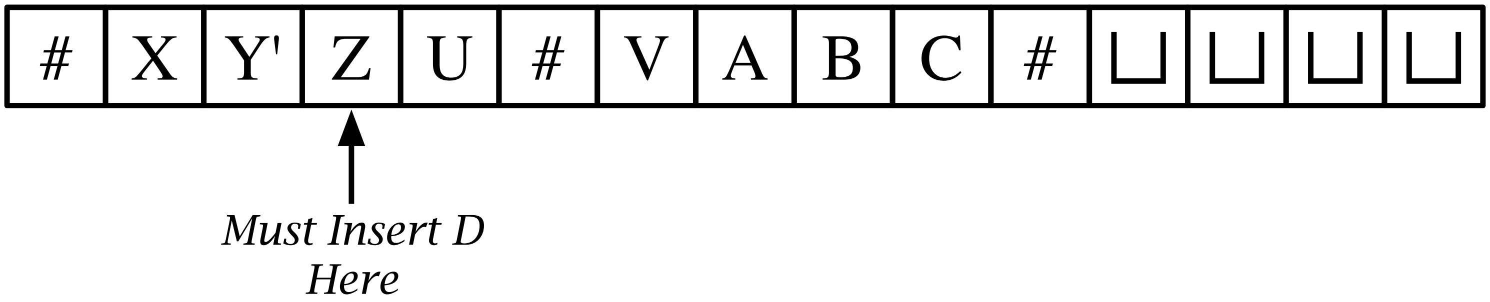
  This triggers insertion:

  - Use a state to remember the symbol to be inserted.
  - Start by memorizing the very first insertion, $\widetilde{\sqcup}$, and then move to the right.
  - While scanning to the right, swap the symbol to be inserted and the symbol stored in the tape cell.
  - Keep scanning until the $\sqcup$ after the very last symbol of the encoding.

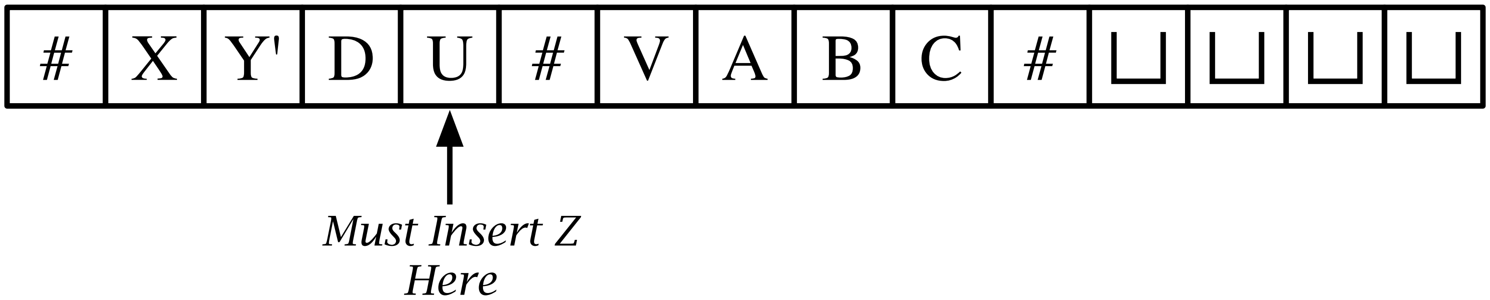
Insertion



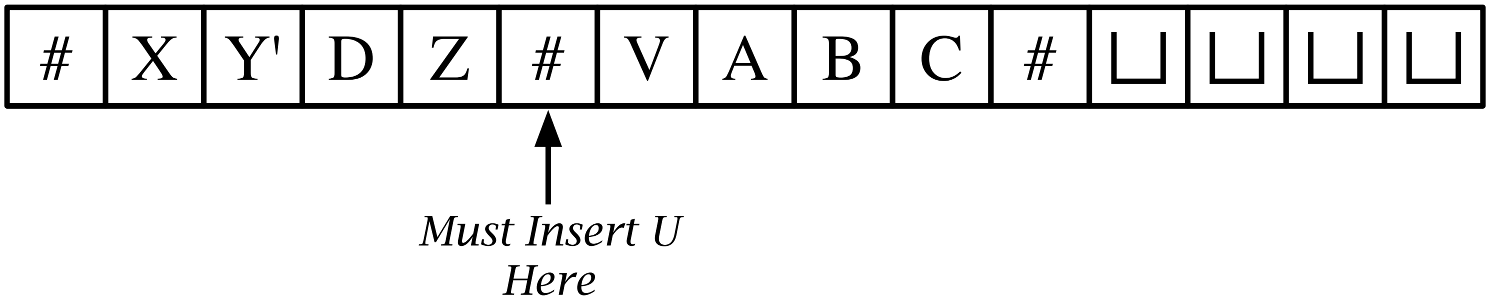
Insertion



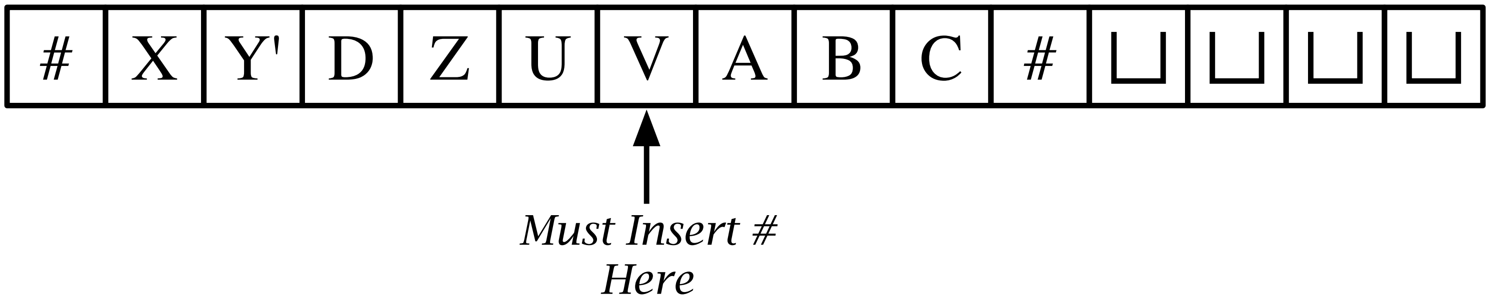
Insertion



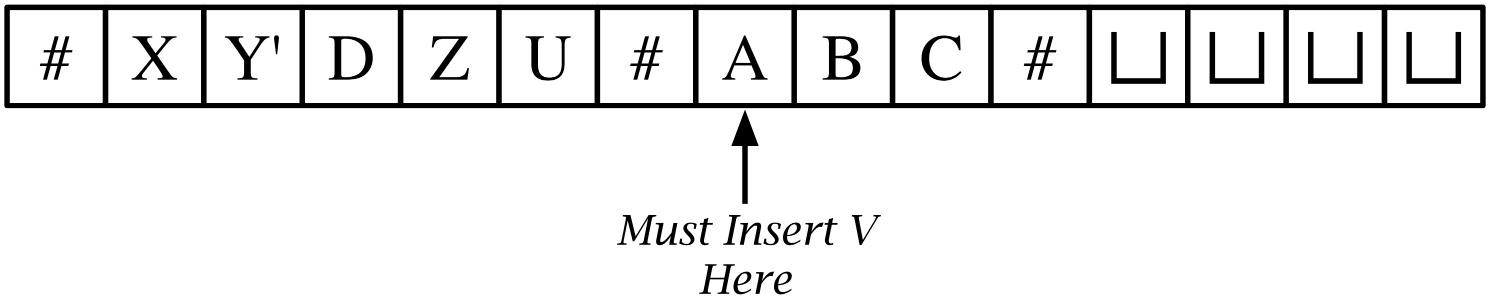
Insertion



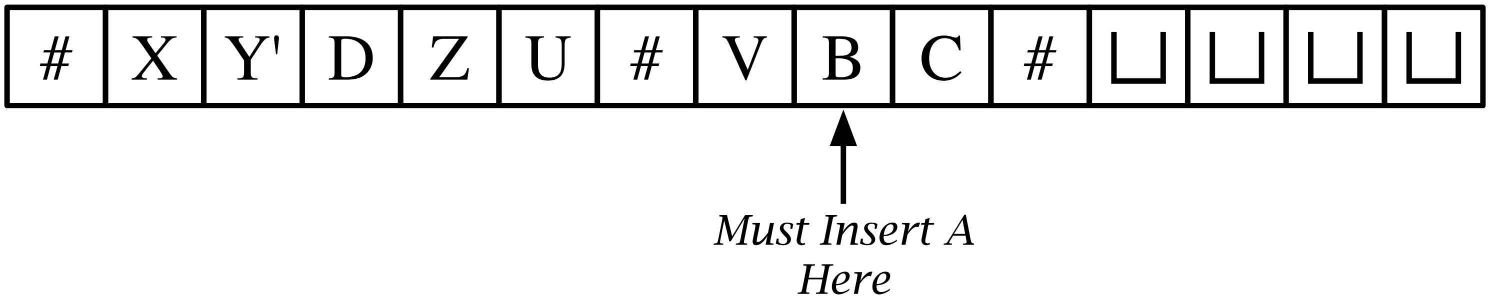
Insertion



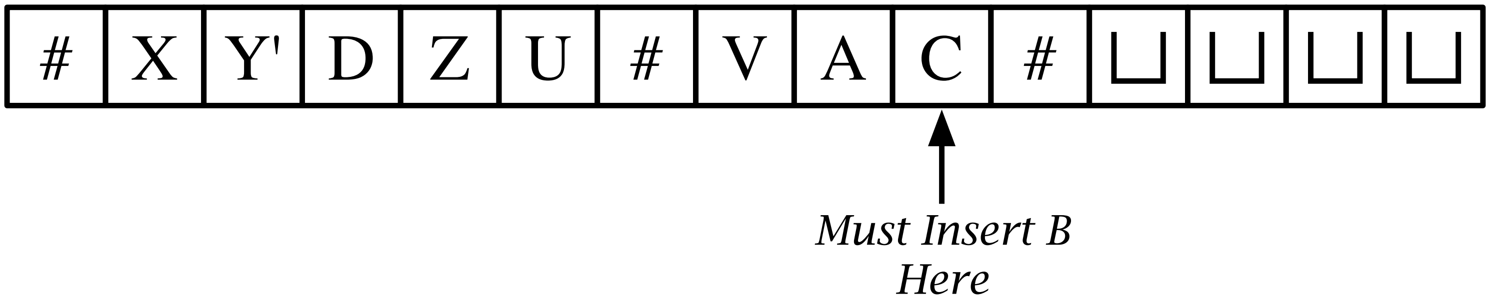
Insertion



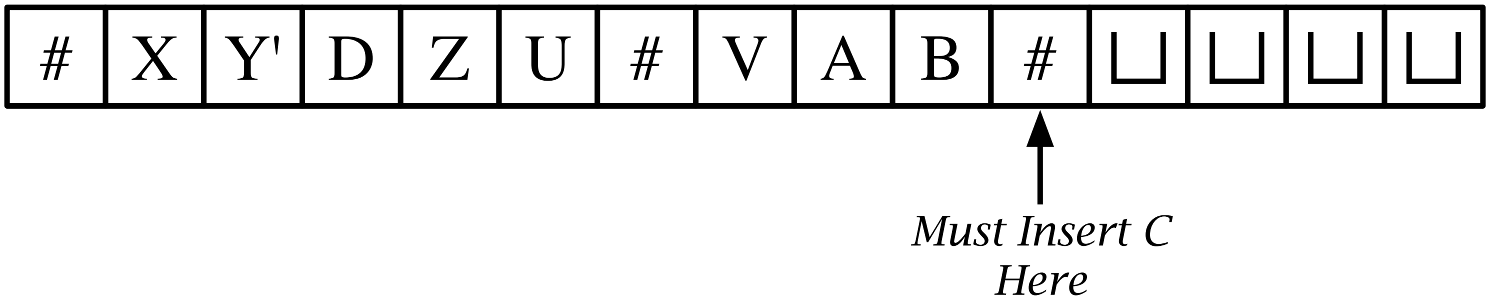
Insertion



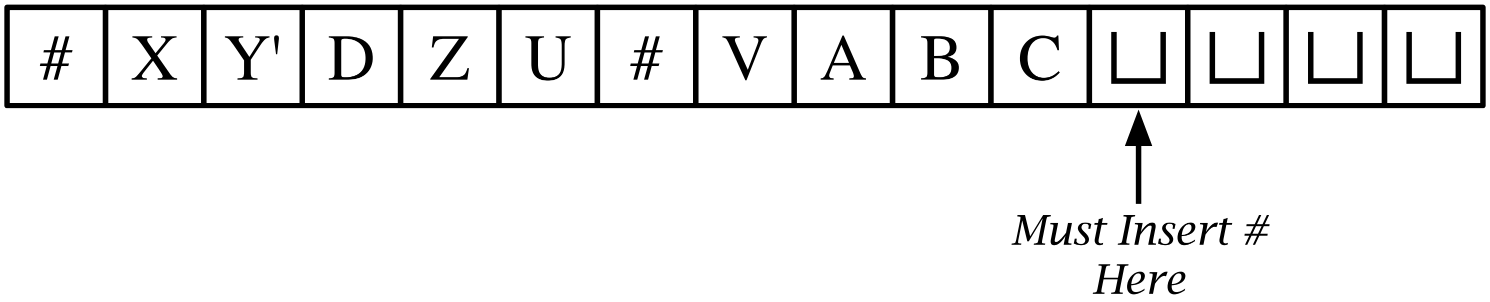
Insertion



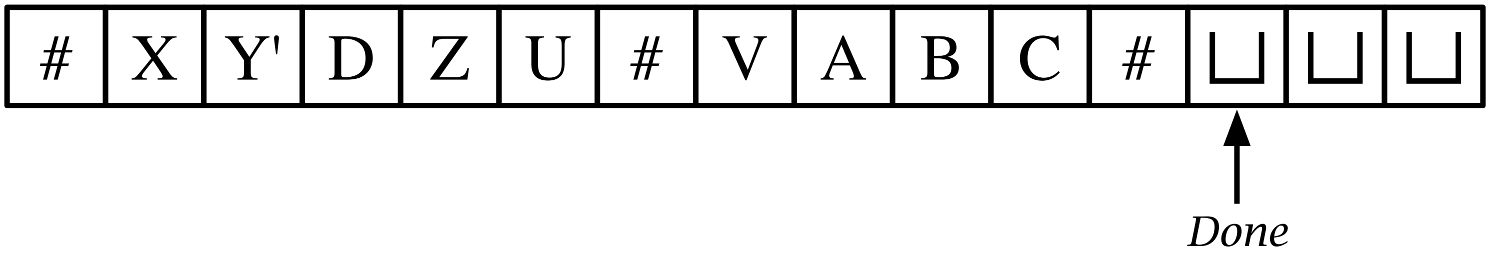
Insertion



Insertion



Insertion



Equivalence Between NTMs and TMs

Equivalence Between NTMs and TMs

Theorem. **Every nondeterministic Turing machine has an equivalent deterministic Turing machine.**

Equivalence Between NTMs and TMs

Theorem. **Every nondeterministic Turing machine has an equivalent deterministic Turing machine.**

Proof We may assume that N is a single-tape machine — we can use the same proof as before.

Equivalence Between NTMs and TMs

Theorem. Every nondeterministic Turing machine has an equivalent deterministic Turing machine.

Proof We may assume that N is a single-tape machine — we can use the same proof as before.

We will construct a three-tape simulator D of N.

Construction

Let C be a constant such that each transition has at most C possible values. Let $\Theta = \{a_1, a_2, \ldots, a_C\}$.

Use a word $p \in \Theta^*$ to **encode a nondeterministic path**, where for all $i \geq 1$ and j, $1 \leq j \leq b$, if the i-th symbol of p is a_j, then it specifies at step i, M must choose the j-th possibility from all possible moves available at that point (if such one exists).

The word p over Θ is a **valid computation path** of N on input w if N on w halts according to the choices written on p.

Three-tape Simulation

Use Tape 1 to store the input, Tape 2 to simulate the tape of N, and Tape 3 to keep an encoding of a computation path.

Define the lexicographic order of paths:
$u_1, \ldots, u_s < v_1, \ldots, v_t \in \Theta^*$ if and only if either

- $s < t$ or
- $s = t$ and there exists some k, $1 \leq k \leq s$, such that $u_1 = v_1, \ldots, u_{k-1} = v_{k-1}$, and $u_k < v_k$.

Here $u_k < v_k$ is evaluated according to a fixed ordering of letters in Θ.

An Algorithm for N

On input w, write the word $\#a_1$ on Tape 3, then repeat:

1. **Copy the input onto Tape 2.**
2. **Try to simulate N on w using the word in Tape 3** as the path. If successful and if N has accepted, then accept and halt.
3. **Modify the path to the next smallest path by incrementing it.**
4. **Erase Tape 2.**

Some Additional Results

Corollary. **A language is Turing-recognizable if and only if it is recognized by a multitape TM.**

Corollary. **A language is Turing-recognizable if and only if it is recognized by an NTM.**

Enumerators

An **enumerator** of a language A is a TM with a special **output tape** such that the machines write on the output tape all the members of A with a special symbol $\#$ as a delimiter.

Enumerators

An **enumerator** of a language A is a TM with a special **output tape** such that the machines write on the output tape all the members of A with a special symbol $\#$ as a delimiter.

Theorem. A language is Turing-recognizable if and only if it has an enumerator.

Enumerators

An **enumerator** of a language A is a TM with a special **output tape** such that the machines write on the output tape all the members of A with a special symbol $\#$ as a delimiter.

Theorem. A language is Turing-recognizable if and only if it has an enumerator.

Proof The "if" part: Simulate the enumerator, and accept when the input word is produced by the enumerator.

The "only if" part: Simulate a recognizer R. For $i = 1, 2, \ldots$, for each w of lexicographic order of at most i, simulate M on w for i steps and outputs w if M accepts w in i steps. ∎

Description of Objects

We assume that there is a systematic way of describing computing devices as well as their inputs. For example, a Turing machine M can be described by putting down in symbols states, symbols, and transition. We fix such an encoding system. We will use $\langle M \rangle$ to represent the encoding of M.

Description of Multiple Objects

To encode multiple objects in a sequence, we simply concatenate the encodings of the objects in order with a special delimiter in between.

Description of Multiple Objects

To encode multiple objects in a sequence, we simply concatenate the encodings of the objects in order with a special delimiter in between.

Special Requirement For Turing machines M and N, $\langle M\rangle\langle N\rangle$ is a representation of a Turing machine that executes M's program first and when M accepts immediately jumps into N's program.

This concept will be used in Chapter 6.