Nonregular Languages

How can we show that a language is not regular?

The Pumping Lemma

Theorem. **(Pumping Lemma)** **Let L be an arbitrary regular language. Then there exists a positive integer p with the following property:**

Given an arbitrary member w of L having length at least p (i.e., $|w| \geq p$), w can be divided into three parts, $w = xyz$, such that

- $|y| \geq 1$ **(the middle part is nonempty),**
- $|xy| \leq p$ **(the first two parts together have length at most p), and**
- **for each $i \geq 0$, $xy^iz \in L$ (removing or repeating the middle part produces members of L).**

Proof of the Pumping Lemma

Let L be an arbitrary regular language. Then there is an FA, say M, that decides L. Let p be the number of states of M.

Let w be an arbitrary member of L having length n with $n \geq p$.

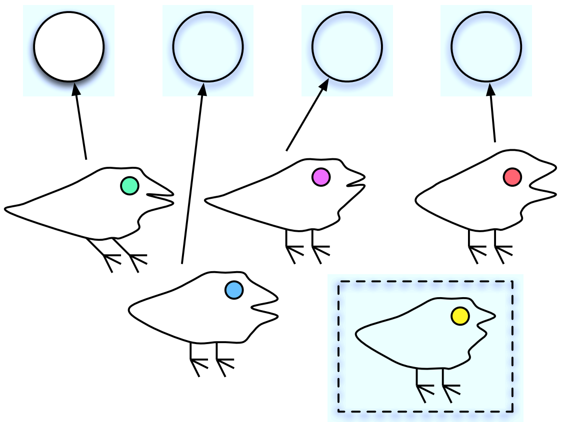
Let $q_0, q_1, \ldots, q_n$ be the states that M on input w. That is, for each i, after reading the first i symbols of w, M is at q_i.

Clearly, q_0 is the initial state of M. Also, because $w \in L$, q_n is a final state of M.

The Pigeonhole Principle

We are placing a number of pigeons in a number of holes.

If there are more pigeons than there are holes, at least one hole should host more than one pigeon.



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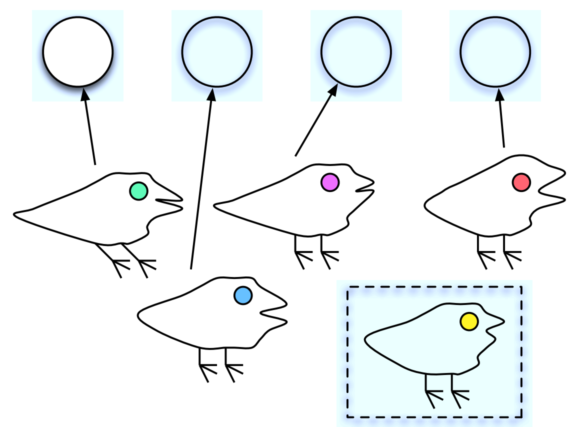

Consider $q_0, \ldots, q_p$ (the first $p+1$ states that M goes through on input w). By the pigeonhole principle, there exist c and d, $0 \leq c < d \leq p$, such that $q_c = q_d$.

Pick an arbitrary such pair (c, d).

Proof of the Pumping Lemma (cont'd)

Let $x = w_1 \cdots w_c$, $y = w_{c+1} \cdots w_d$, and $z = w_{d+1} \cdots w_n$. Then

- $|y| \geq 1$,
- $|xy| \leq p$,
- M transitions from q_0 to q_c on x,
- M transitions from q_c to q_c on y,
- M transitions from q_c to q_n on z.

Thus, for every $i \geq 0$, M transitions from q_0 to q_n on xy^iz, and so xy^iz is a member of L. ∎

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- for each $i \geq 0$, $xy^i z \in B$.

Since $|xy| \leq p$, both x and y consist solely of 0s. The word $xyyz$ has more 0s than 1s, and thus, not in B. However, by the pumping lemma, $xyyz \in B$, a contradiction. Hence, B is not regular. ∎

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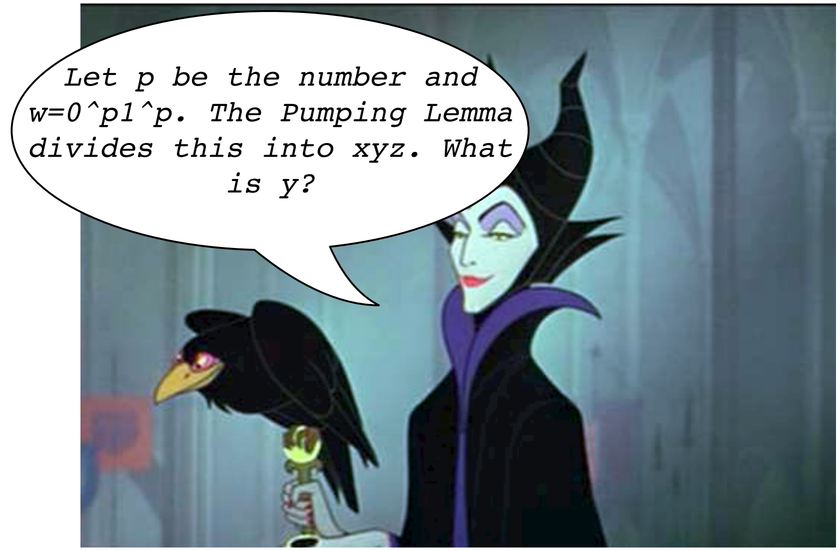

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Let $w' = xz$. Then $w' \in C$ but w' has fewer 0s than 1s. ∎

Example 3

The language $F = \{vv \mid v \in \{0,1\}^*\}$ is not regular (F is the language of all even length strings over $\{0,1\}$ whose first half is identical to the second half).

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Let $w = 0^p 1^p 0^p 1^p$. Then, w is divided into $w = xyz$ such that $|y| > 0$, $|xy| \le p$, and $(\forall i \ge 0)[xy^i z \in F]$. Here $y \in 0^*$ since w begins with 0^p.

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Pick $i = 0$, we have $0^q 1^p 0^p 1^p \in F$, where $q < p$. This word cannot be decomposed as uu. This is a contradiction. ∎

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Let $w = 1^{p^2}$. Then $w = xyz$ for some x, y, z such that $|y| \geq 1$, $|xy| \leq p$, and $(\forall i \geq 0)[xy^i z \in D]$.

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Let $l = |y|$. Then $1 \leq l \leq p$. By plugging in $i = 2$, we have $1^{p^2+l} \in D$, but $p^2 + l \leq p^2 + p < (p+1)^2$, a contradiction. ∎

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With $i = 0$, we have $0^q 1^{p-1} \in E$, where $q \leq p-1$, a contradiction.
∎