Motivations

- ▷ Last ten years: impressive advance in boolean reasoning techniques (SAT)
  - extremely efficient solvers [96, 82, 14, 55, 61, 97]
  - hard "real-world" problems encoded into SAT (e.g.,
    - planning [52, 51, 30, 38],
    - model checking [19, 15, 1, 88, 94, 58, 23, 22, 20, 80],
    - circuit testing [85]
    - security & criptanalysis [57]
    - ...

Motivations (cont.)

- ▷ Recent years: using SAT solvers as boolean reasoning kernels for more expressive solvers
  - combine a SAT reasoner with a domain-specific solver
  - various domains:
    - Modal & description logics [41, 42, 48, 43, 37],
    - temporal reasoning [3],
    - resource planning [95],
    - verification of timed & hybrid systems [60, 6, 9, 84, 29, 65, 70, 8],
    - HW verification [21, 89, 29],
    - SW verification [21, 89],
    - reasoning in combined theories [62, 63, 81, 31, 7, 6, 12, 13, 89, 29, 56, 78, 90, 87, 86]

Content

PART 1:

PROPOSITIONAL SATISFIABILITY

Basics on SAT

Basic notation & definitions

- Boolean formula
  - $\top, \bot$ are formulas
  - A propositional atom $A_1, A_2, A_3, \ldots$ is a formula;
  - if φ_1 and φ_2 are formulas, then $\neg\varphi_1$, $\varphi_1 \wedge \varphi_2$, $\varphi_1 \vee \varphi_2$, $\varphi_1 \rightarrow \varphi_2$, $\varphi_1 \leftrightarrow \varphi_2$ are formulas.
- Literal: a propositional atom A_i (positive literal) or its negation $\neg A_i$ (negative literal)
- N.B.: if $l := \neg A_i$, then $\neg l := A_i$
- $Atoms(\varphi)$: the set $\{A_1, \ldots, A_N\}$ of atoms occurring in φ.
- a boolean formula can be represented as a tree or as a DAG

Semantics of Boolean operators

φ_1	φ_2	$\neg\varphi_1$	$\varphi_1 \wedge \varphi_2$	$\varphi_1 \vee \varphi_2$	$\varphi_1 \rightarrow \varphi_2$	$\varphi_1 \leftrightarrow \varphi_2$
$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\top$	$\top$
$\bot$	$\top$	$\top$	$\bot$	$\top$	$\top$	$\bot$
$\top$	$\bot$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$
$\top$	$\top$	$\bot$	$\top$	$\top$	$\top$	$\top$

N.B.:

$\varphi_1 \vee \varphi_2 := \neg(\neg\varphi_1 \wedge \neg\varphi_2)$,
$\varphi_1 \rightarrow \varphi_2 := (\neg\varphi_1 \vee \varphi_2)$,
$\varphi_1 \leftrightarrow \varphi_2 := (\varphi_1 \rightarrow \varphi_2) \wedge (\varphi_2 \rightarrow \varphi_1)$.

TREE and DAG representation of formulas: example

$$(A_1 \leftrightarrow A_2) \leftrightarrow (A_3 \leftrightarrow A_4)$$

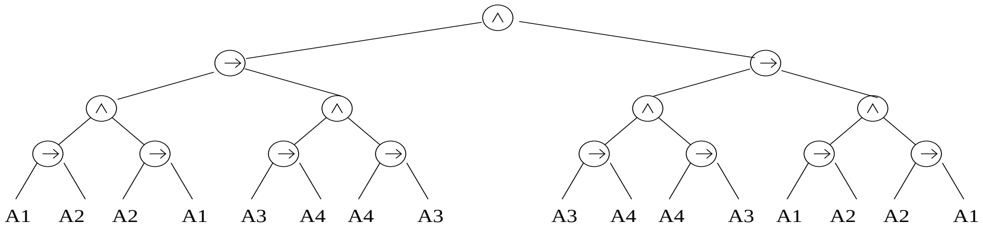
$$\Downarrow$$

$$(((A_1 \leftrightarrow A_2) \rightarrow (A_3 \leftrightarrow A_4)) \wedge$$
$$((A_3 \leftrightarrow A_4) \rightarrow (A_1 \leftrightarrow A_2)))$$

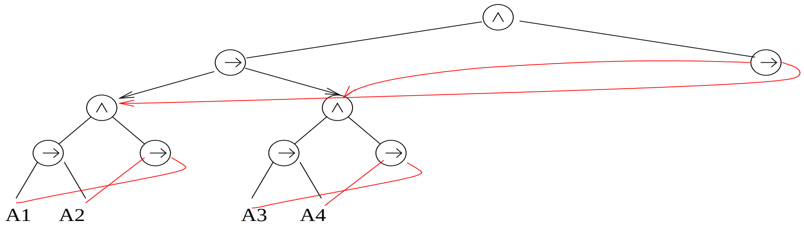
$$\Downarrow$$

$$(((A_1 \rightarrow A_2) \wedge (A_2 \rightarrow A_1)) \rightarrow ((A_3 \rightarrow A_4) \wedge (A_4 \rightarrow A_3))) \wedge$$
$$(((A_3 \rightarrow A_4) \wedge (A_4 \rightarrow A_3)) \rightarrow (((A_1 \rightarrow A_2) \wedge (A_2 \rightarrow A_1))))$$

TREE and DAG representation of formulas: example (cont)



Tree Representation



DAG Representation

Basic notation & definitions (cont)

- Total truth assignment μ for φ:
  $\mu : Atoms(\varphi) \longmapsto \{\top, \bot\}$.
- Partial Truth assignment μ for φ:
  $\mu : \mathcal{A} \longmapsto \{\top, \bot\}$, $\mathcal{A} \subset Atoms(\varphi)$.
- Set and formula representation of an assignment:
  - μ can be represented as a set of literals:
    EX: $\{\mu(A_1) := \top, \mu(A_2) := \bot\} \implies \{A_1, \neg A_2\}$
  - μ can be represented as a formula:
    EX: $\{\mu(A_1) := \top, \mu(A_2) := \bot\} \implies A_1 \wedge \neg A_2$

Basic notation & definitions (cont)

- $\mu \models \varphi$ (μ satisfies φ):
  - $\mu \models A_i \Longleftrightarrow \mu(A_i) = \top$
  - $\mu \models \neg\varphi \Longleftrightarrow$ *not* $\mu \models \varphi$
  - $\mu \models \varphi_1 \wedge \varphi_2 \Longleftrightarrow \mu \models \varphi_1$ *and* $\mu \models \varphi_2$
  - ...
- φ is satisfiable iff $\mu \models \varphi$ for some μ
- $\varphi_1 \models \varphi_2$ (φ_1 entails φ_2):
  $\varphi_1 \models \varphi_2$ iff for every μ $\mu \models \varphi_1 \Longrightarrow \mu \models \varphi_2$
- $\models \varphi$ (φ is valid):
  $\models \varphi$ iff for every μ $\mu \models \varphi$
- φ is valid $\Longleftrightarrow$ $\neg\varphi$ is not satisfiable

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Equivalence and equi-satisfiability

- φ_1 and φ_2 are equivalent iff, for every μ,
  $\mu \models \varphi_1$ iff $\mu \models \varphi_2$
- φ_1 and φ_2 are equi-satisfiable iff
  exists μ_1 s.t. $\mu_1 \models \varphi_1$ iff exists μ_2 s.t. $\mu_2 \models \varphi_2$
- φ_1, φ_2 equivalent

  $$\Downarrow \quad \not\Uparrow$$

  φ_1, φ_2 equi-satisfiable

- EX: $\varphi_1 \vee \varphi_2$ and $(\varphi_1 \vee \neg A_3) \wedge (A_3 \vee \varphi_2)$, A_3 not in $\varphi_1 \vee \varphi_2$,
  are equi-satisfiable but not equivalent.

Complexity

- The problem of deciding the satisfiability of a propositional formula is NP-complete [24].
- The most important logical problems (validity, inference, entailment, equivalence, ...) can be straightforwardly reduced to satisfiability, and are thus (co)NP-complete.

$$\Downarrow$$

No existing worst-case-polynomial algorithm.

NNF, CNF and conversions

POLARITY of subformulas

Polarity: the number of nested negations modulo 2.

- Positive/negative occurrences
  - φ occurs positively in φ;
  - if $\neg\varphi_1$ occurs positively [negatively] in φ, then φ_1 occurs negatively [positively] in φ
  - if $\varphi_1 \wedge \varphi_2$ or $\varphi_1 \vee \varphi_2$ occur positively [negatively] in φ, then φ_1 and φ_2 occur positively [negatively] in φ;
  - if $\varphi_1 \rightarrow \varphi_2$ occurs positively [negatively] in φ, then φ_1 occurs negatively [positively] in φ and φ_2 occurs positively [negatively] in φ;
  - if $\varphi_1 \leftrightarrow \varphi_2$ occurs in φ, then φ_1 and φ_2 occur positively and negatively in φ;

Negative normal form (NNF)

- φ is in Negative normal form iff it is given only by applications of $\wedge, \vee$ to literals.
- every φ can be reduced into NNF:
  1. substituting all $\rightarrow$'s and $\leftrightarrow$'s:

$$\begin{array}{lcl} \varphi_1 \rightarrow \varphi_2 & \Longrightarrow & \neg\varphi_1 \vee \varphi_2 \\ \varphi_1 \leftrightarrow \varphi_2 & \Longrightarrow & (\neg\varphi_1 \vee \varphi_2) \wedge (\varphi_1 \vee \neg\varphi_2) \end{array}$$

  2. pushing down negations recursively:

$$\begin{array}{lcl} \neg(\varphi_1 \wedge \varphi_2) & \Longrightarrow & \neg\varphi_1 \vee \neg\varphi_2 \\ \neg(\varphi_1 \vee \varphi_2) & \Longrightarrow & \neg\varphi_1 \wedge \neg\varphi_2 \\ \neg\neg\varphi_1 & \Longrightarrow & \varphi_1 \end{array}$$

- The reduction is linear if a DAG representation is used.
- Preserves the equivalence of formulas.

NNF: example

$$(A_1 \leftrightarrow A_2) \leftrightarrow (A_3 \leftrightarrow A_4)$$

$$\Downarrow$$

$$((((A_1 \to A_2) \wedge (A_1 \leftarrow A_2)) \to ((A_3 \to A_4) \wedge (A_3 \leftarrow A_4))) \wedge$$
$$(((A_1 \to A_2) \wedge (A_1 \leftarrow A_2)) \leftarrow ((A_3 \to A_4) \wedge (A_3 \leftarrow A_4))))$$

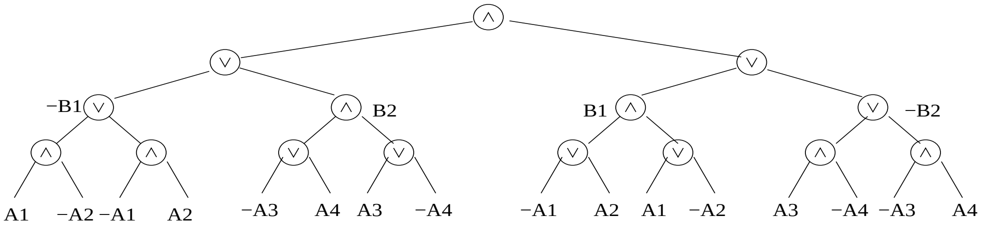
$$\Downarrow$$

$$((\neg((\neg A_1 \vee A_2) \wedge (A_1 \vee \neg A_2)) \vee ((\neg A_3 \vee A_4) \wedge (A_3 \vee \neg A_4))) \wedge$$
$$(((\neg A_1 \vee A_2) \wedge (A_1 \vee \neg A_2)) \vee \neg((\neg A_3 \vee A_4) \wedge (A_3 \vee \neg A_4))))$$

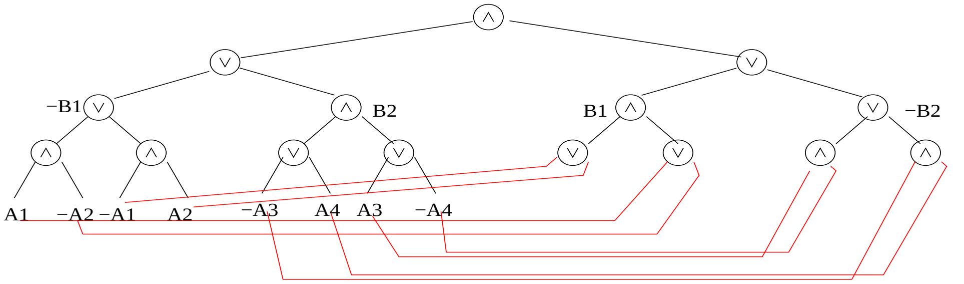
$$\Downarrow$$

$$((((A_1 \wedge \neg A_2) \vee (\neg A_1 \wedge A_2)) \vee ((\neg A_3 \vee A_4) \wedge (A_3 \vee \neg A_4))) \wedge$$
$$(((\neg A_1 \vee A_2) \wedge (A_1 \vee \neg A_2)) \vee ((A_3 \wedge \neg A_4) \vee (\neg A_3 \wedge A_4))))$$

NNF: example (cont)



Tree Representation



DAG Representation

N.B. For each non-literal subformula φ, φ and ¬φ have different representations ⟹ they are not shared.

Conjunctive Normal Form (CNF)

- φ is in Conjunctive normal form iff it is a conjunction of disjunctions of literals:

$$\bigwedge_{i=1}^{L} \bigvee_{j_i=1}^{K_i} l_{j_i}$$

- the disjunctions of literals $\bigvee_{j_i=1}^{K_i} l_{j_i}$ are called clauses
- Easier to handle: list of lists of literals.

  $\Longrightarrow$ no reasoning on the recursive structure of the formula

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Classic CNF Conversion $CNF(\varphi)$

- Every φ can be reduced into CNF by, e.g.,
  1. converting it into NNF;
  2. applying recursively the DeMorgan's Rule:

$$(\varphi_1 \wedge \varphi_2) \vee \varphi_3 \quad \Longrightarrow \quad (\varphi_1 \vee \varphi_3) \wedge (\varphi_2 \vee \varphi_3)$$

- Worst-case exponential.
- $Atoms(CNF(\varphi)) = Atoms(\varphi)$.
- $CNF(\varphi)$ is equivalent to φ.
- Normal: if φ_1 equivalent to φ_2, then $CNF(\varphi_1)$ identical to $CNF(\varphi_2)$ modulo reordering.
- Rarely used in practice.

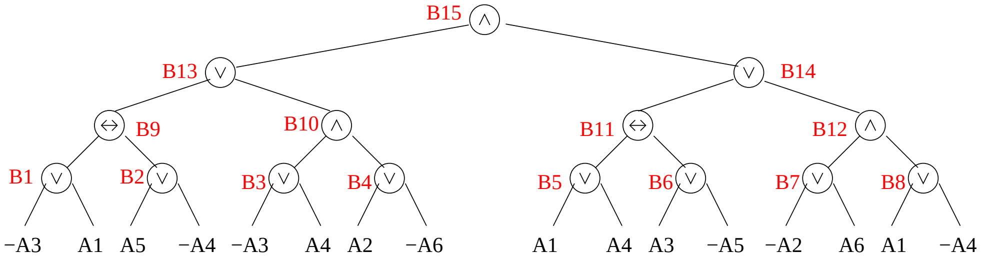
Labeling CNF conversion $CNF_{label}(\varphi)$ [71, 28]

- Every φ can be reduced into CNF by, e.g., applying recursively bottom-up the rules:

$$\varphi \implies \varphi[(l_i \vee l_j)|B] \wedge CNF(B \leftrightarrow (l_i \vee l_j))$$
$$\varphi \implies \varphi[(l_i \wedge l_j)|B] \wedge CNF(B \leftrightarrow (l_i \wedge l_j))$$
$$\varphi \implies \varphi[(l_i \leftrightarrow l_j)|B] \wedge CNF(B \leftrightarrow (l_i \leftrightarrow l_j))$$

  l_i, l_j being literals and B being a "new" variable.
- Worst-case linear.
- $Atoms(CNF_{label}(\varphi)) \supseteq Atoms(\varphi)$.
- $CNF_{label}(\varphi)$ is equi-satisfiable w.r.t. φ.
- Non-normal.
- More used in practice.

Labeling CNF conversion CNF_{label} – example



$CNF(B_1 \leftrightarrow (\neg A_3 \vee A_1))$ $\wedge$
$\ldots$ $\wedge$
$CNF(B_8 \leftrightarrow (A_1 \vee \neg A_4))$ $\wedge$
$CNF(B_9 \leftrightarrow (B_1 \leftrightarrow B_2))$ $\wedge$
$\ldots$ $\wedge$
$CNF(B_{12} \leftrightarrow (B_7 \wedge B_8))$ $\wedge$
$CNF(B_{13} \leftrightarrow (B_9 \vee B_{10}))$ $\wedge$
$CNF(B_{14} \leftrightarrow (B_{11} \vee B_{12}))$ $\wedge$
$CNF(B_{15} \leftrightarrow (B_{13} \wedge B_{14}))$ $\wedge$
B_{15}

Labeling CNF conversion CNF_{label} (improved)

- As in the previous case, applying instead the rules:

$$\varphi \Longrightarrow \varphi[(l_i \vee l_j)|B] \wedge CNF(B \rightarrow (l_i \vee l_j)) \ if \ (l_i \vee l_j) \ pos.$$

$$\varphi \Longrightarrow \varphi[(l_i \vee l_j)|B] \wedge CNF((l_i \vee l_j) \rightarrow B) \ if \ (l_i \vee l_j) \ neg.$$

$$\varphi \Longrightarrow \varphi[(l_i \wedge l_j)|B] \wedge CNF(B \rightarrow (l_i \wedge l_j)) \ if \ (l_i \wedge l_j) \ pos.$$

$$\varphi \Longrightarrow \varphi[(l_i \wedge l_j)|B] \wedge CNF((l_i \wedge l_j) \rightarrow B) \ if \ (l_i \wedge l_j) \ neg.$$

$$\varphi \Longrightarrow \varphi[(l_i \leftrightarrow l_j)|B] \wedge CNF(B \rightarrow (l_i \leftrightarrow l_j)) \ if \ (l_i \leftrightarrow l_j) \ pos.$$

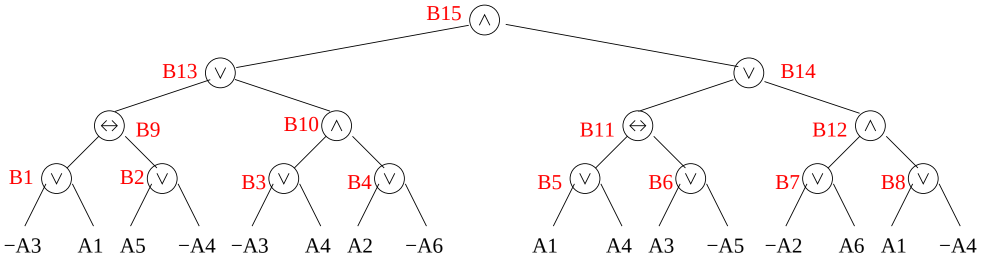
$$\varphi \Longrightarrow \varphi[(l_i \leftrightarrow l_j)|B] \wedge CNF((l_i \leftrightarrow l_j) \rightarrow B) \ if \ (l_i \leftrightarrow l_j) \ neg.$$

- Smaller in size:

$$CNF(B \rightarrow (l_i \vee l_j)) = (\neg B \vee l_i \vee l_j)$$

$$CNF(((l_i \vee l_j) \rightarrow B)) = (\neg l_i \vee B) \wedge (\neg l_j \vee B)$$

Labeling CNF conversion CNF_{label} – example



Basic		Improved	
$CNF(B_1 \leftrightarrow (\neg A_3 \vee A_1))$	$\wedge$	$CNF(B_1 \leftrightarrow (\neg A_3 \vee A_1))$	$\wedge$
$\ldots$	$\wedge$	$\ldots$	$\wedge$
$CNF(B_8 \leftrightarrow (A_1 \vee \neg A_4))$	$\wedge$	$CNF(B_8 \rightarrow (A_1 \vee \neg A_4))$	$\wedge$
$CNF(B_9 \leftrightarrow (B_1 \leftrightarrow B_2))$	$\wedge$	$CNF(B_9 \rightarrow (B_1 \leftrightarrow B_2))$	$\wedge$
$\ldots$	$\wedge$	$\ldots$	$\wedge$
$CNF(B_{12} \leftrightarrow (B_7 \wedge B_8))$	$\wedge$	$CNF(B_{12} \rightarrow (B_7 \wedge B_8))$	$\wedge$
$CNF(B_{13} \leftrightarrow (B_9 \vee B_{10}))$	$\wedge$	$CNF(B_{13} \rightarrow (B_9 \vee B_{10}))$	$\wedge$
$CNF(B_{14} \leftrightarrow (B_{11} \vee B_{12}))$	$\wedge$	$CNF(B_{14} \rightarrow (B_{11} \vee B_{12}))$	$\wedge$
$CNF(B_{15} \leftrightarrow (B_{13} \wedge B_{14}))$	$\wedge$	$CNF(B_{15} \rightarrow (B_{13} \wedge B_{14}))$	$\wedge$
B_{15}		B_{15}	

Basic SAT techniques

Truth Tables

- Exhaustive evaluation of all subformulas:

φ_1	φ_2	$\varphi_1 \wedge \varphi_2$	$\varphi_1 \vee \varphi_2$	$\varphi_1 \rightarrow \varphi_2$	$\varphi_1 \leftrightarrow \varphi_2$
$\bot$	$\bot$	$\bot$	$\bot$	$\top$	$\top$
$\bot$	$\top$	$\bot$	$\top$	$\top$	$\bot$
$\top$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

- Requires polynomial space.
- Never used in practice.

Semantic tableaux [83]

- Search for an assignment satisfying φ
- applies recursively elimination rules to the connectives
- If a branch contains A_i and $\neg A_i$, (ψ_i and $\neg\psi_1$) for some i, the branch is closed, otherwise it is open.
- if no rule can be applied to an open branch μ, then $\mu \models \varphi$;
- if all branches are closed, the formula is not satisfiable;

Tableau elimination rules

$$\frac{\varphi_1 \wedge \varphi_2}{\begin{array}{c}\varphi_1\\ \varphi_2\end{array}} \qquad \frac{\neg(\varphi_1 \vee \varphi_2)}{\begin{array}{c}\neg\varphi_1\\ \neg\varphi_2\end{array}} \qquad \frac{\neg(\varphi_1 \rightarrow \varphi_2)}{\begin{array}{c}\varphi_1\\ \neg\varphi_2\end{array}} \qquad (\wedge\text{-elimination})$$

$$\frac{\neg\neg\varphi}{\varphi} \qquad (\neg\neg\text{-elimination})$$

$$\frac{\varphi_1 \vee \varphi_2}{\varphi_1 \quad \varphi_2} \qquad \frac{\neg(\varphi_1 \wedge \varphi_2)}{\neg\varphi_1 \quad \neg\varphi_2} \qquad \frac{\varphi_1 \rightarrow \varphi_2}{\neg\varphi_1 \quad \varphi_2} \qquad (\vee\text{-elimination})$$

$$\frac{\varphi_1 \leftrightarrow \varphi_2}{\begin{array}{cc}\varphi_1 & \neg\varphi_1\\ \varphi_2 & \neg\varphi_2\end{array}} \qquad \frac{\neg(\varphi_1 \leftrightarrow \varphi_2)}{\begin{array}{cc}\varphi_1 & \neg\varphi_1\\ \neg\varphi_2 & \varphi_2\end{array}} \qquad (\leftrightarrow\text{-elimination}).$$

Semantic Tableaux – example

$$\varphi = (A_1 \vee A_2) \wedge (A_1 \vee \neg A_2) \wedge (\neg A_1 \vee A_2) \wedge (\neg A_1 \vee \neg A_2)$$

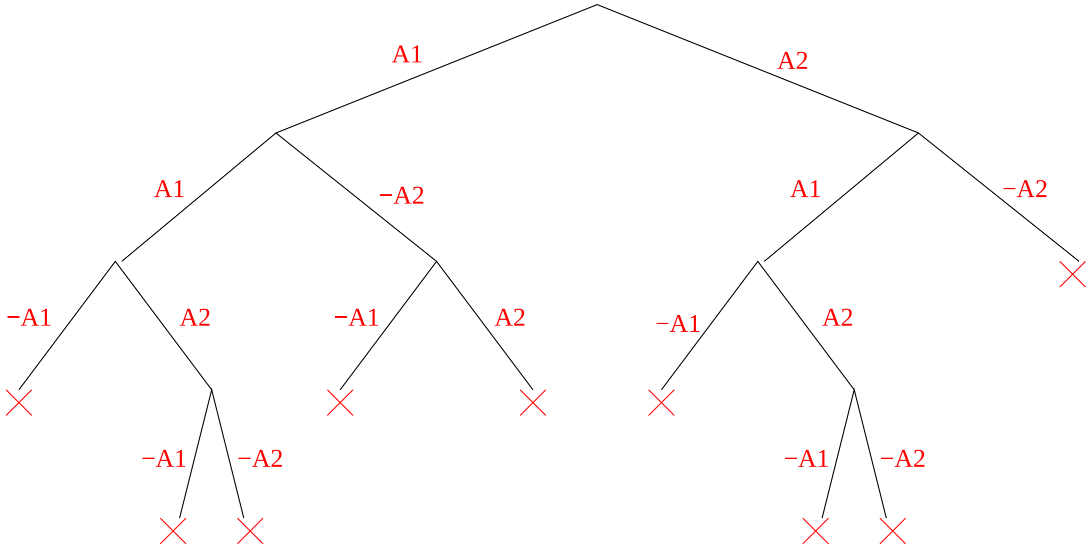


Tableau algorithm

```
function Tableau(Γ)
    if $A_i \in \Gamma$ and $\neg A_i \in \Gamma$                          /* branch closed */
        then return False;
    if $(\varphi_1 \wedge \varphi_2) \in \Gamma$                            /* ∧-elimination */
        then return Tableau($\Gamma \cup \{\varphi_1, \varphi_2\} \setminus \{(\varphi_1 \wedge \varphi_2)\}$);
    if $(\neg\neg\varphi_1) \in \Gamma$                                     /* ¬¬-elimination */
        then return Tableau($\Gamma \cup \{\varphi_1\} \setminus \{(\neg\neg\varphi_1)\}$);
    if $(\varphi_1 \vee \varphi_2) \in \Gamma$                              /* ∨-elimination */
        then return    Tableau($\Gamma \cup \{\varphi_1\} \setminus \{(\varphi_1 \vee \varphi_2)\}$) or
                       Tableau($\Gamma \cup \{\varphi_2\} \setminus \{(\varphi_1 \vee \varphi_2)\}$);
    ...
    return True;                                                            /* branch expanded */
```

Semantic Tableaux – summary

- Handles all propositional formulas (CNF not required).
- Branches on disjunctions
- Intuitive, modular, easy to extend

  $\Longrightarrow$ loved by logicians.
- Rather inefficient

  $\Longrightarrow$ avoided by computer scientists.
- Requires polynomial space

DPLL [27, 26]

- Davis-Putnam-Longeman-Loveland procedure (DPLL)
- Tries to build recursively an assignment μ satisfying φ;
- At each recursive step assigns a truth value to (all instances of) one atom.
- Performs deterministic choices first.

DPLL rules

$$\frac{\varphi_1 \wedge (l)}{\varphi_1[l|\top]} \ (Unit)$$

$$\frac{\varphi}{\varphi[l|\top]} \ (l\ Pure)$$

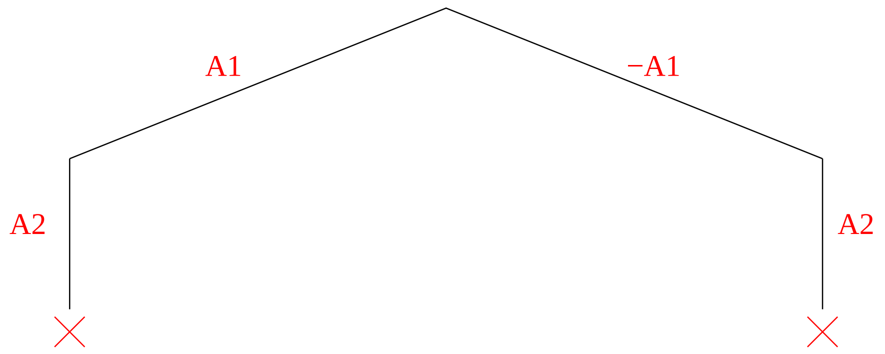
$$\frac{\varphi}{\varphi[l|\top] \quad \varphi[l|\bot]} \ (split)$$

(l is a pure literal in φ iff it occurs only positively).

- Split applied if and only if the others cannot be applied.
- Equivalent formalism described in [90]

DPLL – example

$$\varphi = (A_1 \vee A_2) \wedge (A_1 \vee \neg A_2) \wedge (\neg A_1 \vee A_2) \wedge (\neg A_1 \vee \neg A_2)$$



DPLL Algorithm

```
function DPLL($\varphi, \mu$)
    if $\varphi = \top$                                    /* base      */
        then return True;
    if $\varphi = \bot$                                    /* backtrack */
        then return False;
    if {a unit clause $(l)$ occurs in $\varphi$}           /* unit      */
        then return DPLL($assign(l, \varphi), \mu \wedge l$);
    if {a literal $l$ occurs pure in $\varphi$}            /* pure      */
        then return DPLL($assign(l, \varphi), \mu \wedge l$);
    l := choose-literal($\varphi$);                        /* split     */
    return  DPLL($assign(l, \varphi), \mu \wedge l$)  or
            DPLL($assign(\neg l, \varphi), \mu \wedge \neg l$);
```

DPLL – summary

- Handles CNF formulas (non-CNF variant known [5, 40]).
- Branches on truth values
  $\Longrightarrow$all instances of an atom assigned simultaneously
- Postpones branching as much as possible.
- Mostly ignored by logicians.
- Probably the most efficient SAT algorithm
  $\Longrightarrow$ loved by computer scientists.
- Requires polynomial space
- Choose_literal() critical!
- Many very efficient implementations [96, 82, 14, 61].
- A library: SIM [39]

Stalmark's procedure [79]

– Using triplets to represent formulas (represents DAGS)

$$\underbrace{(A_1 \wedge \overbrace{(A_2 \wedge A_3)}^{B_3})}_{B_1} \vee \underbrace{(\neg A_1 \wedge \neg \overbrace{(A_2 \wedge A_3)}^{B_3})}_{B_2}$$

$$\Downarrow$$

$$\begin{gathered}(B_1 \vee B_2) \wedge \\ (B_1 \leftrightarrow A_1 \wedge B_3) \wedge \\ (B_2 \leftrightarrow \neg A_1 \wedge \neg B_3) \wedge \\ (B_3 \leftrightarrow A_2 \wedge A_3)\end{gathered}$$

– Breadth first search up to depth 2

Stalmark's procedure (cont.)

– Try both sides of a branch to find forced decisions. EX:

- $(A \vee B) \wedge (\neg A \vee C) \wedge (\neg A \vee B) \wedge (A \vee D)$
-

$$\begin{array}{ccc} A = \bot & \Longrightarrow & B = \top, D = \top \\ A = \top & \Longrightarrow & B = \top, C = \top \end{array}$$

$$\Downarrow$$

$$B = \top$$

– Repeat for all variables (depth 1) and variable pairs (depth 2)

– if not sufficient, run a DPLL-like procedure on the resulting formula

Stalmark's procedure – summary

- Handles non-CNF formulas in DAG form
- Branches on truth values (of subformulas)
- very efficient with particular kinds of formulas (e.g., circuits)
- Requires polynomial space
- No freely available implementation

Ordered Binary Decision Diagrams (OBDDs) [19]

- Normal representation of a boolean formula.
- "If-then-else" binary DAGs with two leaves: 1 and 0
- Variable ordering $A_1, A_2, ..., A_n$ imposed a priory.
- Paths leading to 1 represent models

  Paths leading to 0 represent counter-models
- Once built, logical operations (satisfiability, validity, equivalence) immediate.
- Finds all models.

(Implicit) OBDD structure

- $OBDD(\top, \{...\}) = 1$,
- $OBDD(\bot, \{...\}) = 0$,
- $OBDD(\varphi, \{A_1, A_2, ..., A_n\}) =$
  $if\ A_1$
  $then\ OBDD(\varphi[A_1|\top], \{A_2, ..., A_n\})$
  $else\ OBDD(\varphi[A_1|\bot], \{A_2, ..., A_n\})$

OBDD - Examples

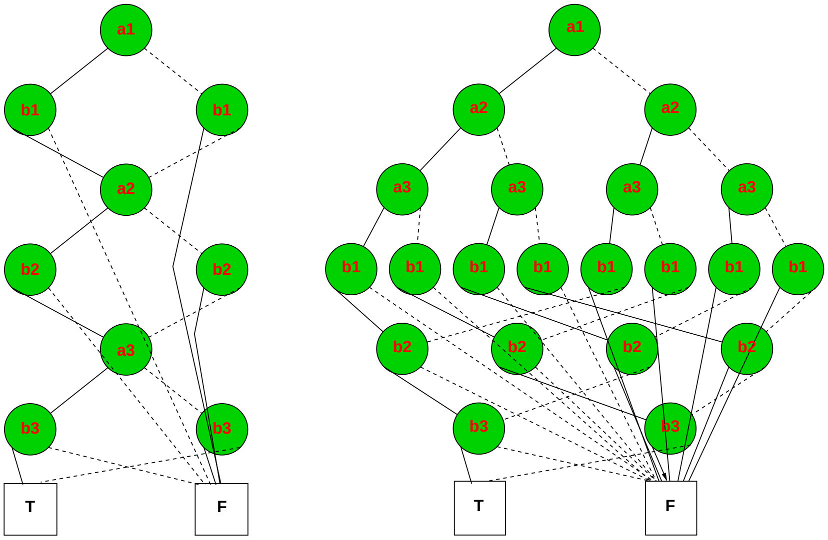


Figure 1: OBDDs of $(a_1 \leftrightarrow b_1) \wedge (a_2 \leftrightarrow b_2) \wedge (a_3 \leftrightarrow b_3)$ with different variable orderings

Incrementally building an OBDD

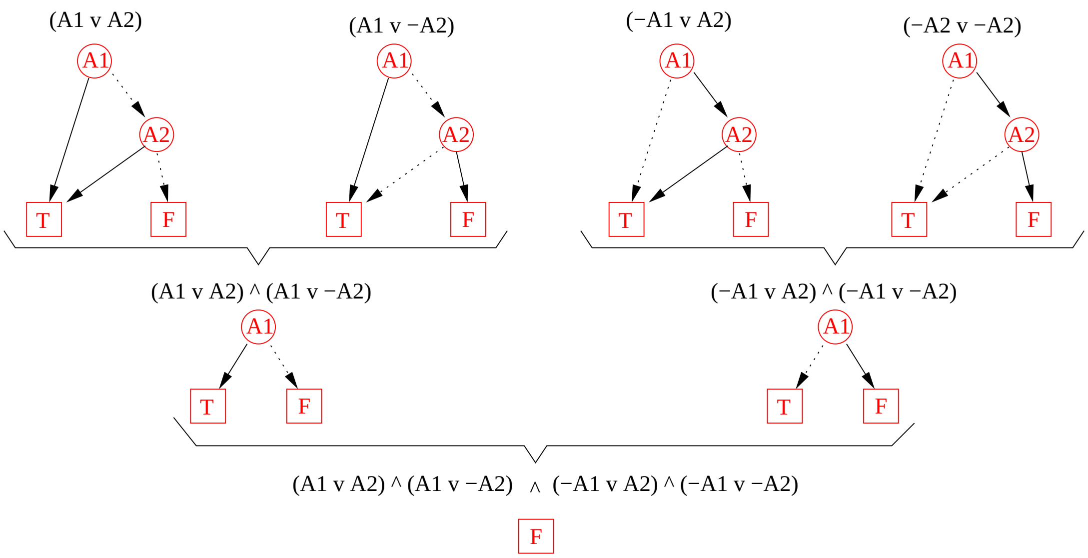
- $obdd\_build(\top, \{...\}) := 1$,
- $obdd\_build(\bot, \{...\}) := 0$,
- $obdd\_build((\varphi_1\ op\ \varphi_2), \{A_1, ..., A_n\}) :=$

$$
\begin{array}{ll}
obdd\_merge(& op, \\
 & obdd\_build(\varphi_1, \{A_1, ..., A_n\}), \\
 & obdd\_build(\varphi_2, \{A_1, ..., A_n\}), \\
 & \{A_1, ..., A_n\} \\
)
\end{array}
$$

$op \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$

OBBD incremental building – example

$$\varphi = (A_1 \vee A_2) \wedge (A_1 \vee \neg A_2) \wedge (\neg A_1 \vee A_2) \wedge (\neg A_1 \vee \neg A_2)$$



OBDD – summary

- Handle all propositional formulas (CNF not required).
- (Implicitly) branch on truth values.
- Find all models.
- Factorize common parts of the search tree (DAG)
- Require setting a variable ordering a priori (critical!)
  $\Longrightarrow$ cannot postpone branching
- Very efficient for some problems (circuits, model checking).
- Require exponential space in worst-case
- Used by Hardware community, ignored by logicians, recently introduced in computer science.

Incomplete SAT techniques: GSAT, WSAT [76, 75]

- Hill-Climbing techniques: GSAT, WSAT
- looks for a complete assignment;
- starts from a random assignment;
- Greedy search: looks for a better “neighbor” assignment
- Avoid local minima: restart & random walk

GSAT algorithm

```
function GSAT(φ)
    for $i := 1$ to Max-tries do
        $\mu$ := rand-assign(φ);
        for $j := 1$ to Max-flips do
            if ($score(\varphi, \mu) = 0$)
                then return True;
                else Best-flips := hill-climb($\varphi, \mu$);
                    $A_i$ := rand-pick(Best-flips);
                    $\mu$ := flip($A_i, \mu$);
        end
    end
    return "no satisfying assignment found".
```

GSAT & WSAT– summary

- Handle only CNF formulas.
- Incomplete
- Extremely efficient for some (satisfiable) problems.
- Require polynomial space
- Used in Artificial Intelligence (e.g., planning)
- Non-CNF Variants: NC-GSAT [73], DAG-SAT [74]

DPLL Heuristics & Optimizations

Techniques to achieve efficiency in DPLL

- Preprocessing: preprocess the input formula so that to make it easier to solve
- Look-ahead: exploit information about the remaining search space
  - unit propagation
  - pure literal
  - forward checking (splitting heuristics)
- Look-back: exploit information about search which has already taken place
  - Backjumping
  - Learning

Variants of DPLL

DPLL is a family of algorithms.

- different splitting heuristics
- preprocessing: (subsumption, 2-simplification)
- backjumping
- learning
- random restart
- horn relaxation
- ...

Iterative description of DPLL [82, 97]

```
status = preprocess();
if (status!=UNKNOWN) return status;
while(1) {
    decide_next_branch();
    while (1) {
        status = deduce();
        if (status == CONFLICT) {
            blevel = analyze_conflict();
            if (blevel == 0)
                return UNSATISFIABLE; else
         backtrack(blevel);
        }
        else if (status == SATISFIABLE)
            return SATISFIABLE;
        else break;
}   }
```

Splitting heuristics - Choose_literal()

- Split is the source of non-determinism for DPLL
- Choose_literal() critical for efficiency
- many split heuristics conceived in literature.

Some example heuristics

- MOMS heuristics: pick the literal occurring most often in the minimal size clauses

  ⟹fast and simple
- Jeroslow-Wang: choose the literal with maximum

  $$score(l) := \Sigma_{l \in c \ \& \ c \in \varphi} \ 2^{-|c|}$$

  ⟹estimates l's contribution to the satisfiability of φ
- Satz [55]: selects a candidate set of literals, perform unit propagation, chooses the one leading to smaller clause set

  ⟹maximizes the effects of unit propagation
- Chaff's VSIDS [61]: variable state independent decaying sum
- ...

Some preprocessing techniques

- Sorting+subsumption:

$$\varphi_1 \wedge (l_2 \vee l_1) \wedge \varphi_2 \wedge (l_2 \vee l_3 \vee l_1) \wedge \varphi_3$$

$$\Downarrow$$

$$\varphi_1 \wedge (l_1 \vee l_2) \wedge \varphi_2 \wedge \varphi_3$$

Some preprocessing techniques (cont.)

- 2-simplify [17]: exploiting binary clauses.
- **Repeat:**
  1. build the implication graph induced by literals
  2. detect strongly connected cycles
     $\Longrightarrow$equivalence classes of literals
  3. perform substitutions
  4. perform unit and pure.

  Until no more simplification possible.
- Very useful for many application domains.
- Improvement: Hypre [11].1

Conflict-directed backtracking (backjumping) [14, 82]

- Idea: when a branch fails,
  1. reveal the sub-assignment causing the failure (conflict set)
  2. backtrack to the most recent branching point in the conflict set
- a conflict set is constructed from the conflict clause by tracking backwards the unit-implications causing it and by keeping the branching literals.
- when a branching point fails, a conflict set is obtained by resolving the two conflict sets of the two branches.
- may avoid lots of redundant search.

Conflict-directed backtracking – example

$\neg A_1 \vee A_2$

$\neg A_1 \vee A_3 \vee A_9$

$\neg A_2 \vee \neg A_3 \vee A_4$

$\neg A_4 \vee A_5 \vee A_{10}$

$\neg A_4 \vee A_6 \vee A_{11}$

$\neg A_5 \vee \neg A_6$

$A_1 \vee A_7 \vee \neg A_{12}$

$A_1 \vee A_8$

$\neg A_7 \vee \neg A_8 \vee \neg A_{13}$

$\ldots$

Conflict-directed backtracking – example (cont.)

$\neg A_1 \vee A_2$

$\neg A_1 \vee A_3 \vee A_9$

$\neg A_2 \vee \neg A_3 \vee A_4$

$\neg A_4 \vee A_5 \vee A_{10}$

$\neg A_4 \vee A_6 \vee A_{11}$

$\neg A_5 \vee \neg A_6$

$A_1 \vee A_7 \vee \neg A_{12}$

$A_1 \vee A_8$

$\neg A_7 \vee \neg A_8 \vee \neg A_{13}$

$\ldots$

$\{\ldots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \ldots\}$ (initial assignment)

Conflict-directed backtracking – example (cont.)

$\neg A_1 \vee A_2$

$\neg A_1 \vee A_3 \vee A_9$

$\neg A_2 \vee \neg A_3 \vee A_4$

$\neg A_4 \vee A_5 \vee A_{10}$

$\neg A_4 \vee A_6 \vee A_{11}$

$\neg A_5 \vee \neg A_6$

$A_1 \vee A_7 \vee \neg A_{12}$ $\quad$ $true \Longrightarrow removed$

$A_1 \vee A_8$ $\quad$ $true \Longrightarrow removed$

$\neg A_7 \vee \neg A_8 \vee \neg A_{13}$

$\ldots$

$\{\ldots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \ldots, A_1\}$ (branch on A_1)

Conflict-directed backtracking – example (cont.)

$\neg A_1 \vee A_2$ $\quad$ *true* $\Longrightarrow$ *removed*

$\neg A_1 \vee A_3 \vee A_9$ $\quad$ *true* $\Longrightarrow$ *removed*

$\neg A_2 \vee \neg A_3 \vee A_4$

$\neg A_4 \vee A_5 \vee A_{10}$

$\neg A_4 \vee A_6 \vee A_{11}$

$\neg A_5 \vee \neg A_6$

$A_1 \vee A_7 \vee \neg A_{12}$ $\quad$ *true* $\Longrightarrow$ *removed*

$A_1 \vee A_8$ $\quad$ *true* $\Longrightarrow$ *removed*

$\neg A_7 \vee \neg A_8 \vee \neg A_{13}$

$\ldots$

$\{\ldots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \ldots, A_1, A_2, A_3\}$
(unit A_2, A_3)

Conflict-directed backtracking – example (cont.)

$\neg A_1 \vee A_2$ $\quad$ *true* $\Longrightarrow$ *removed*

$\neg A_1 \vee A_3 \vee A_9$ $\quad$ *true* $\Longrightarrow$ *removed*

$\neg A_2 \vee \neg A_3 \vee A_4$ $\quad$ *true* $\Longrightarrow$ *removed*

$\neg A_4 \vee A_5 \vee A_{10}$

$\neg A_4 \vee A_6 \vee A_{11}$

$\neg A_5 \vee \neg A_6$

$A_1 \vee A_7 \vee \neg A_{12}$ $\quad$ *true* $\Longrightarrow$ *removed*

$A_1 \vee A_8$ $\quad$ *true* $\Longrightarrow$ *removed*

$\neg A_7 \vee \neg A_8 \vee \neg A_{13}$

$\ldots$

$\{\ldots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \ldots, A_1, A_2, A_3, A_4\}$
(unit A_4)

Conflict-directed backtracking – example (cont.)

$\neg A_1 \vee A_2$	$true \Longrightarrow removed$
$\neg A_1 \vee A_3 \vee A_9$	$true \Longrightarrow removed$
$\neg A_2 \vee \neg A_3 \vee A_4$	$true \Longrightarrow removed$
$\neg A_4 \vee A_5 \vee A_{10}$	$true \Longrightarrow removed$
$\neg A_4 \vee A_6 \vee A_{11}$	$true \Longrightarrow removed$
$\neg A_5 \vee \neg A_6$	$false \Longrightarrow conflict$
$A_1 \vee A_7 \vee \neg A_{12}$	$true \Longrightarrow removed$
$A_1 \vee A_8$	$true \Longrightarrow removed$
$\neg A_7 \vee \neg A_8 \vee \neg A_{13}$	
$\ldots$	

$\{\ldots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \ldots, A_1, A_2, A_3, A_4, A_5, A_6\}$
(unit A_5, A_6)

Conflict-directed backtracking – example (cont.)

$\neg A_1 \vee A_2$	$true \Longrightarrow removed$
$\neg A_1 \vee A_3 \vee A_9$	$true \Longrightarrow removed$
$\neg A_2 \vee \neg A_3 \vee A_4$	$true \Longrightarrow removed$
$\neg A_4 \vee A_5 \vee A_{10}$	$true \Longrightarrow removed$
$\neg A_4 \vee A_6 \vee A_{11}$	$true \Longrightarrow removed$
$\neg A_5 \vee \neg A_6$	$false \Longrightarrow conflict$
$A_1 \vee A_7 \vee \neg A_{12}$	$true \Longrightarrow removed$
$A_1 \vee A_8$	$true \Longrightarrow removed$
$\neg A_7 \vee \neg A_8 \vee \neg A_{13}$	
$\ldots$	

$\Longrightarrow$Conflict set: $\{\neg A_9, \neg A_{10}, \neg A_{11}, A_1\}$ $\Longrightarrow$backtrack to A_1

Conflict-directed backtracking – example (cont.)

$\neg A_1 \vee A_2 \qquad true \Longrightarrow removed$

$\neg A_1 \vee A_3 \vee A_9 \qquad true \Longrightarrow removed$

$\neg A_2 \vee \neg A_3 \vee A_4$

$\neg A_4 \vee A_5 \vee A_{10}$

$\neg A_4 \vee A_6 \vee A_{11}$

$\neg A_5 \vee \neg A_6$

$A_1 \vee A_7 \vee \neg A_{12}$

$A_1 \vee A_8$

$\neg A_7 \vee \neg A_8 \vee \neg A_{13}$

$\ldots$

$\{\ldots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \ldots, \neg A_1\}$ (branch on $\neg A_1$)

Conflict-directed backtracking – example (cont.)

$\neg A_1 \vee A_2 \qquad true \Longrightarrow removed$

$\neg A_1 \vee A_3 \vee A_9 \qquad true \Longrightarrow removed$

$\neg A_2 \vee \neg A_3 \vee A_4$

$\neg A_4 \vee A_5 \vee A_{10}$

$\neg A_4 \vee A_6 \vee A_{11}$

$\neg A_5 \vee \neg A_6$

$A_1 \vee A_7 \vee \neg A_{12} \qquad true \Longrightarrow removed$

$A_1 \vee A_8 \qquad true \Longrightarrow removed$

$\neg A_7 \vee \neg A_8 \vee \neg A_{13} \; false \Longrightarrow conflict$

$\ldots$

$\{\ldots, \neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}, \ldots, \neg A_1, A_7, A_8\}$

(unit A_7, A_8)

Conflict-directed backtracking – example (cont.)

$\neg A_1 \lor A_2 \quad \mathit{true} \Longrightarrow \mathit{removed}$

$\neg A_1 \lor A_3 \lor A_9 \quad \mathit{true} \Longrightarrow \mathit{removed}$

$\neg A_2 \lor \neg A_3 \lor A_4$

$\neg A_4 \lor A_5 \lor A_{10}$

$\neg A_4 \lor A_6 \lor A_{11}$

$\neg A_5 \lor \neg A_6$

$A_1 \lor A_7 \lor \neg A_{12} \quad \mathit{true} \Longrightarrow \mathit{removed}$

$A_1 \lor A_8 \quad \mathit{true} \Longrightarrow \mathit{removed}$

$\neg A_7 \lor \neg A_8 \lor \neg A_{13} \quad \mathit{false} \Longrightarrow \mathit{conflict}$

$\ldots$

$\Longrightarrow$conflict set: $\{A_{12}, A_{13}, \neg A_1\}$.

Conflict-directed backtracking – example (cont.)

$\neg A_1 \lor A_2 \quad \mathit{true} \Longrightarrow \mathit{removed}$

$\neg A_1 \lor A_3 \lor A_9 \quad \mathit{true} \Longrightarrow \mathit{removed}$

$\neg A_2 \lor \neg A_3 \lor A_4$

$\neg A_4 \lor A_5 \lor A_{10}$

$\neg A_4 \lor A_6 \lor A_{11}$

$\neg A_5 \lor \neg A_6$

$A_1 \lor A_7 \lor \neg A_{12} \quad \mathit{true} \Longrightarrow \mathit{removed}$

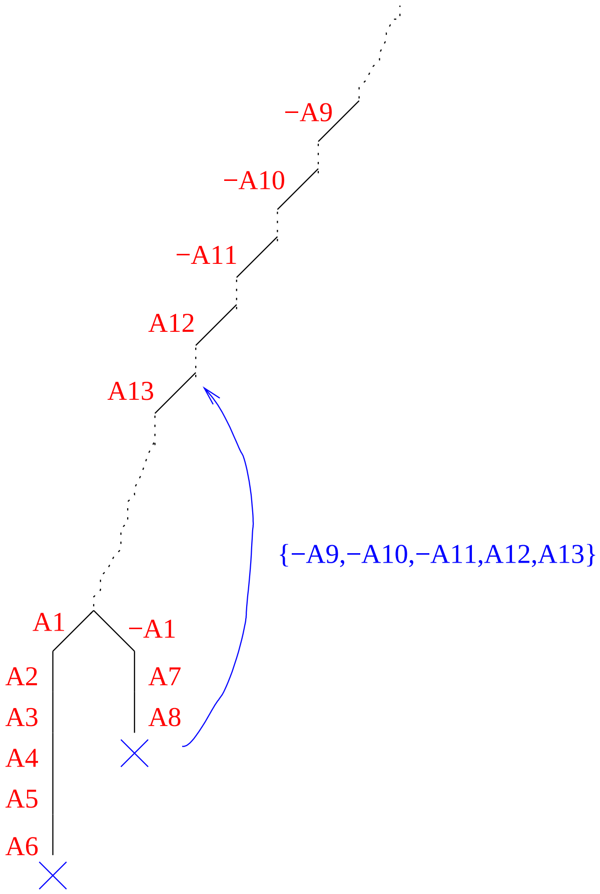
$A_1 \lor A_8 \quad \mathit{true} \Longrightarrow \mathit{removed}$

$\neg A_7 \lor \neg A_8 \lor \neg A_{13} \quad \mathit{false} \Longrightarrow \mathit{conflict}$

$\ldots$

$\Longrightarrow$conflict set: $\{A_{12}, A_{13}, \neg A_1\} \ldots \lor \{\neg A_9, \neg A_{10}, \neg A_{11}, A_1\}$

$\Longrightarrow \{\neg A_9, \neg A_{10}, \neg A_{11}, A_{12}, A_{13}\} \Longrightarrow$ backtrack to A_{13}.



Learning [14, 82]

- Idea: When a conflict set C is revealed, then $\neg C$ can be added to the clause set
  $\Longrightarrow$DPLL will never again generate an assignment containing C.
- May avoid a lot of redundant search.
- Problem: may cause a blowup in space
  $\Longrightarrow$techniques to control learning and to drop learned clauses when necessary

Learning – example (cont.)

$\neg A_1 \vee A_2 \quad true \Longrightarrow removed$

$\neg A_1 \vee A_3 \vee A_9 \quad true \Longrightarrow removed$

$\neg A_2 \vee \neg A_3 \vee A_4 \quad true \Longrightarrow removed$

$\neg A_4 \vee A_5 \vee A_{10} \quad true \Longrightarrow removed$

$\neg A_4 \vee A_6 \vee A_{11} \quad true \Longrightarrow removed$

$\neg A_5 \vee \neg A_6 \quad false \Longrightarrow conflict$

$A_1 \vee A_7 \vee \neg A_{12} \quad true \Longrightarrow removed$

$A_1 \vee A_8 \quad true \Longrightarrow removed$

$\neg A_7 \vee \neg A_8 \vee \neg A_{13}$

$\ldots$

$A_9 \vee A_{10} \vee A_{11} \vee \neg A_1 \quad learned\ clause$

$\Longrightarrow$Conflict set: $\{\neg A_9, \neg A_{10}, \neg A_{11}, A_1\}$

$\Longrightarrow$learn $A_9 \vee A_{10} \vee A_{11} \vee \neg A_1$

PART 2:

BEYOND PROPOSITIONAL SATISFIABILITY

Goal

Integrate SAT procedures with domain-specific solvers in an efficient way

Different viewpoints:

- (Logicians' communities) Provide a new "SAT based" general framework from which to build efficient decision procedures (alternative, e.g., to semantic tableaux)
- (SAT community) Extend SAT techniques to more expressive domains (preserving efficiency)
- (Decision procedures community) Optimize the boolean component of reasoning

Key issues

- Correctness, completeness & termination
  - A general logic framework
  - A general integration schema
- Efficiency
  - Efficiency issues of the SAT procedure
  - Efficiency issues of the domain-specific solver
  - Efficiency of the integration

    $\Longrightarrow$A procedure integrating a very efficient SAT solver with a very efficient domain-specific solver may be dramatically inefficient if the integration is not done properly.

Formal Framework

Ingredients

– A logic language $\mathcal{L}$ extending boolean logic:
  - Language-specific atomic expression are formulas (e.g., $P(x)$, $\Box(A_1 \vee \Box A_2)$, $(x - y \geq 6)$, $\exists\ \textsc{Children}\ (\textsc{male} \wedge \textsc{teen})$)
  - if φ_1 and φ_2 formulas, then $\neg\varphi_1$, $\varphi_1 \wedge \varphi_2$, $\varphi_1 \vee \varphi_2$, $\varphi_1 \rightarrow \varphi_2$, $\varphi_1 \leftrightarrow \varphi_2$ are formulas.
  - Nothing else is a formula (e.g., no external quantifiers!)

Ingredients (cont.)

– A semantic for $\mathcal{L}$ extending standard boolean one:

$$\begin{array}{lcl}
M \models \psi, (\psi\ atomic) & \Longleftrightarrow & \text{[definition specific for } \mathcal{L}\text{]} \\
M \models \neg\phi & \Longleftrightarrow & M \not\models \phi \\
M \models \varphi_1 \wedge \varphi_2 & \Longleftrightarrow & M \models \varphi_1 \textbf{ and } M \models \varphi_2 \\
M \models \varphi_1 \vee \varphi_2 & \Longleftrightarrow & M \models \varphi_1 \textbf{ or } M \models \varphi_2 \\
M \models \varphi_1 \rightarrow \varphi_2 & \Longleftrightarrow & \textbf{if}M \models \varphi_1 \textbf{ then } M \models \varphi_2 \\
M \models \varphi_1 \leftrightarrow \varphi_2 & \Longleftrightarrow & M \models \varphi_1 \textbf{ iff } M \models \varphi_2
\end{array}$$

Ingredients (cont.)

– A language-specific procedure $\mathcal{L}$-SOLVE able to decide the satisfiability of lists of atomic expressions and their negations E.g.:

- FO-SOLVE($\{P(x,a), P(b,y)\}$) $\Longrightarrow$Sat
- K-SOLVE($\{\Box(A_1 \rightarrow A_2), \Box(A_1), \neg\Box(A_2)\}$) $\Longrightarrow$Unsat
- MATH-SOLVE($\{(x-y \leq 3), (y-z \leq 4), \neg(x-z \leq 8)\}$) $\Longrightarrow$Unsat
- $\mathcal{ALC}$-SOLVE $$\left(\left\{\begin{array}{l} \forall\,\textsc{Children}\ (\neg\textsc{male} \vee \textsc{teen}), \\ \forall\,\textsc{Children}\ (\textsc{male}), \\ \exists\,\textsc{Children}\ (\neg\ \textsc{teen}) \end{array}\right\}\right)$$ $\Longrightarrow$Unsat

Definitions: atoms, literals

- An atom is every formula in $\mathcal{L}$ whose main connective is not a boolean operator.
- A literal is either an atom (a positive literal) or its negation (a negative literal).
- Examples:
  $P(x),\ \neg\forall x.Q(x, f(a))$
  $\Box(A_1 \vee \Box A_2),\ \neg\Box(A_2 \rightarrow \Box(A_3 \vee A_4))$
  $(x - y \geq 6),\ \neg(z - y < 2),$
  $\exists\ \textsc{Children}\ (\textsc{male} \wedge \textsc{teen}), \neg\forall\ \textsc{Parent}\ (\textsc{old})$
- $Atoms(\varphi)$: the set of top-level atoms in φ.

Definitions: total truth assignment

- We call a total truth assignment μ for φ a total function

$$\mu : Atoms(\varphi) \longmapsto \{\top, \bot\}$$

- We represent a total truth assignment μ either as a set of literals

$$\mu = \{\alpha_1, \ldots, \alpha_N, \neg\beta_1, \ldots, \neg\beta_M, A_1, \ldots, A_R, \neg A_{R+1}, \ldots, \neg A_S\},$$

or as a boolean formula

$$\mu = \bigwedge_i \alpha_i \wedge \bigwedge_j \neg\beta_j \wedge \bigwedge_{k=1}^{R} A_k \wedge \bigwedge_{h=R+1}^{S} \neg A_h$$

Definitions: partial truth assignment

- We call a partial truth assignment μ for φ a partial function

$$\mu : Atoms(\varphi) \longmapsto \{\top, \bot\}$$

- Partial truth assignments can be represented as sets of literals or as boolean functions, as before.
- A partial truth assignment μ for φ is a subset of a total truth assignment for φ.
- If $\mu_2 \subseteq \mu_1$, then we say that μ_1 extends μ_2 and that μ_2 subsumes μ_1.
- a conflict set for μ_1 is an inconsistent subset $\mu_2 \subseteq \mu_1$ s.t. no strict subset of μ_2 is inconsistent.

Definitions: total and partial truth assignment (cont.)

Remark:

- Syntactically identical instances of the same atom in φ are always assigned identical truth values.
  E.g., $... \wedge ((t_1 \geq t_2) \vee A_1) \wedge ((t_1 \geq t_2) \vee A_2) \wedge ...$
- Equivalent but syntactically different atoms in φ may (in principle) be assigned different truth values.
  E.g., $... \wedge ((t_1 \geq t_2) \vee A_1) \wedge ((t_2 \leq t_1) \vee A_2) \wedge ...$

Definition: propositional satisfiability in $\mathcal{L}$

A truth assignment μ for φ propositionally satisfies φ in $\mathcal{L}$, written $\mu \models_p \varphi$, iff it makes φ evaluate to $\top$:

$$\begin{array}{lll}
\mu \models_p \varphi_1,\ \varphi_1 \in Atoms(\varphi) & \Longleftrightarrow & \varphi_1 \in \mu; \\
\mu \models_p \neg\varphi_1 & \Longleftrightarrow & \mu \not\models_p \varphi_1; \\
\mu \models_p \varphi_1 \wedge \varphi_2 & \Longleftrightarrow & \mu \models_p \varphi_1 \textit{ and } \mu \models_p \varphi_2. \\
\ldots & \ldots & \ldots
\end{array}$$

– A partial assignment μ propositionally satisfies φ iff all total assignments extending μ propositionally satisfy φ.

Definition: propositional satisfiability in $\mathcal{L}$ (cont)

– Intuition: If φ is seen as a boolean combination of its atoms, $\models_p$ is standard propositional satisfiability.

– Atoms seen as (recognizable) blackboxes

– The definitions of $\varphi_1 \models_p \varphi_2$, $\models_p \varphi$ is straightforward.

– $\models_p$ stronger than $\models$: if $\varphi_1 \models_p \varphi_2$, then $\varphi_1 \models \varphi_2$, but not vice versa.
E.g., $(v_1 \le v_2) \wedge (v_2 \le v_3) \models (v_1 \le v_3)$, but
$(v_1 \le v_2) \wedge (v_2 \le v_3) \not\models_p (v_1 \le v_3)$.

Satisfiability and propositional satisfiability in $\mathcal{L}$

Proposition: φ is satisfiable in $\mathcal{L}$ iff there exists a truth assignment μ for φ s.t.

- $\mu \models_p \varphi$, and
- μ is satisfiable in $\mathcal{L}$.

– Search decomposed into two orthogonal components:

- Purely propositional: search for a truth assignments μ propositionally satisfying φ
- Purely domain-dependent: verify the satisfiability in $\mathcal{L}$ of μ.

Example

$$
\begin{aligned}
\varphi = \quad & \{\neg(2v_2 - v_3 > 2) \lor A_1\} \land \\
& \{\neg A_2 \lor (2v_1 - 4v_5 > 3)\} \land \\
& \{(3v_1 - 2v_2 \leq 3) \lor A_2\} \land \\
& \{\neg(2v_3 + v_4 \geq 5) \lor \neg(3v_1 - v_3 \leq 6) \lor \neg A_1\} \land \\
& \{A_1 \lor (3v_1 - 2v_2 \leq 3)\} \land \\
& \{(v_1 - v_5 \leq 1) \lor (v_5 = 5 - 3v_4) \lor \neg A_1\} \land \\
& \{A_1 \lor (v_3 = 3v_5 + 4) \lor A_2\}.
\end{aligned}
$$

$$\mu = \{\neg(2v_2 - v_3 > 2), \neg A_2, (3v_1 - 2v_2 \leq 3), (v_1 - v_5 \leq 1), \neg(3v_1 - v_3 \leq 6), (v_3 = 3v_5 + 4)\}.$$

$$\mu' = \{\neg(2v_2 - v_3 > 2), \neg A_2, \neg A_1, (3v_1 - 2v_2 \leq 3), (v_3 = 3v_5 + 4)\}.$$

– $\mu \models_p \varphi$, but is unsatisfiable, as contains conflict sets:

$$
\begin{aligned}
& \{(3v_1 - 2v_2 \leq 3), \neg(2v_2 - v_3 > 2), \neg(3v_1 - v_3 \leq 6)\} \\
& \{(v_1 - v_5 \leq 1), (v_3 = 3v_5 + 4), \neg(3v_1 - v_3 \leq 6)\}
\end{aligned}
$$

– $\mu' \models_p \varphi$, and is satisfiable ($v_1, v_2, v_3 := 0,\ v_5 := -4/3$).

Complete collection of assignments

A collection $\mathcal{M} = \{\mu_1, \ldots, \mu_n\}$ of (possibly partial) assignments propositionally satisfying φ is complete iff

$$\models_p \varphi \leftrightarrow \bigvee_j \mu_j. \qquad (1)$$

- for every total assignment η s.t. $\eta \models_p \varphi$, there is $\mu_i \in \mathcal{M}$ s.t. $\mu_i \subseteq \eta$.
  $\Longrightarrow \mathcal{M}$ represents all assignments.
- $\mathcal{M}$ “compact” representation of the whole set of total assignments propositionally satisfying φ.

Complete collection of assignments and satisfiability in $\mathcal{L}$

Proposition. Let $\mathcal{M} = \{\mu_1, \ldots, \mu_n\}$ be a complete collection of truth assignments propositionally satisfying φ. Then φ is satisfiable if and only if μ_j is satisfiable for some $\mu_j \in \mathcal{M}$.

- Search decomposed into two orthogonal components:
  - Purely propositional: generate (in a lazy way) a complete collection $\mathcal{M} = \{\mu_1, \ldots, \mu_n\}$ of truth assignments propositionally satisfying φ;
  - Purely domain-dependent: check one by one the satisfiability in $\mathcal{L}$ of the μ_i's.

Redundancy of complete collection of assignments

A complete collection $\mathcal{M} = \{\mu_1, \ldots, \mu_n\}$ of assignments propositionally satisfying φ is

- strongly non redundant iff, for every $\mu_i, \mu_j \in \mathcal{M}$, $(\mu_i \wedge \mu_j)$ is propositionally unsatisfiable,
- non redundant iff, for every $\mu_j \in \mathcal{M}$, $\mathcal{M} \setminus \{\mu_j\}$ is no more complete,
- redundant otherwise.

- If $\mathcal{M}$ is redundant, then $\mu_j \supseteq \mu_i$ for some $\mu_i, \mu_j \in \mathcal{M}$:

$$\begin{array}{lcll}
\models_p \varphi \leftrightarrow \bigvee_{i \neq j} \mu_i & \Longrightarrow & \models_p \bigvee_i \mu_i \leftrightarrow \bigvee_{i \neq j} \mu_i & \Longrightarrow \\
\bigvee_i \mu_i \models_p \bigvee_{i \neq j} \mu_i & \Longrightarrow & \mu_j \models_p \bigvee_{i \neq j} \mu_i & \Longrightarrow \\
\mu_j \models_p \mu_i \;\; for\ some\ i & \Longrightarrow & \mu_j \supseteq \mu_i &
\end{array}$$

- If $\mathcal{M}$ is strongly non redundant, then $\mathcal{M}$ is non redundant:

$$\begin{array}{ll}
\mu_j \wedge \mu_i \;\; propositionally\ inconsistent & \Longrightarrow \\
\mu_j \models_p \neg \mu_i & \Longrightarrow \\
\mathcal{M}\ non\ redundant &
\end{array}$$

Redundancy: example

Let $\varphi := (\alpha \vee \beta \vee \gamma) \wedge (\alpha \vee \beta \vee \neg\gamma)$, α, β, γ atoms. Then

1. $\{\{\alpha,\beta,\gamma\},\{\alpha,\beta,\neg\gamma\},\{\alpha,\neg\beta,\gamma\},\{\alpha,\neg\beta,\neg\gamma\},$ $\{\neg\alpha,\beta,\gamma\},\{\neg\alpha,\beta,\neg\gamma\}\}$ is the set of all total assignments propositionally satisfying φ;
2. $\{\{\alpha\},\{\alpha,\beta\},\{\alpha,\neg\gamma\},\{\alpha,\beta\},\{\beta\},\{\beta,\neg\gamma\},\{\alpha,\gamma\},\{\beta,\gamma\}\}$ is complete but redundant;
3. $\{\{\alpha\},\{\beta\}\}$ is complete, non redundant but not strongly non redundant;
4. $\{\{\alpha\},\{\neg\alpha,\beta\}\}$ is complete and strongly non redundant.

A Generalized Search Procedure

Truth assignment enumerator

A truth assignment enumerator is a total function $\textsc{Assign\_Enumerator}()$ which takes as input a formula φ in $\mathcal{L}$ and returns a complete collection $\{\mu_1, \ldots, \mu_n\}$ of assignments propositionally satisfying φ.

– A truth assignment enumerator is
  - strongly non-redundant if $\textsc{Assign\_Enumerator}(\varphi)$ is strongly non-redundant, for every φ,
  - non-redundant if $\textsc{Assign\_Enumerator}(\varphi)$ is non-redundant, for every φ,
  - redundant otherwise.

Truth assignment enumerator w.r.t. SAT solver

Remark. Notice the difference:

– A SAT solver has to find only one satisfying assignment —or to decide there is none;

– A Truth assignment enumerator has to find a complete collection of satisfying assignments.

A generalized procedure

```
boolean $\mathcal{L}$-SAT($formula\ \varphi, assignment\ \&\ \mu, model\ \&\ M$)
    do
        $\mu :=$ NEXT_ASSIGNMENT($\varphi$)                /* next in $\{\mu_1, ..., \mu_n\}$ */
        if ($\mu \neq Null$)
            $satifiable := \mathcal{L}$-SOLVE($\mu, M$);
    while ((satifiable = False) and ($\mu \neq Null$))
    if ($satifiable \neq$ False)
    then return True;                               /* a satisf. assignment found */
    else return False;                              /* no satisf. assignment found */
```

$\mathcal{L}$-SAT

- $\mathcal{L}$-SAT(φ) terminating, correct and complete $\Longleftrightarrow$ $\mathcal{L}$-SOLVE(μ) terminating, correct and complete.
- $\mathcal{L}$-SAT depends on $\mathcal{L}$ only for $\mathcal{L}$-SOLVE
- $\mathcal{L}$-SAT requires polynomial space iff
  - $\mathcal{L}$-SOLVE requires polynomial space and
  - ASSIGN_ENUMERATOR is lazy

Mandatory requirements for an assignment enumerator

An assignment enumerator must always:

- (Termination) terminate
- (Correctness) generate assignments propositionally satisfying φ
- (Completeness) generate complete set of assignments

Mandatory requirements for $\mathcal{L}$-SOLVE()

$\mathcal{L}$-SOLVE() must always:

- (Termination) terminate
- (Correctness & completeness) return $True$ if μ is satisfiable in $\mathcal{L}$, $False$ otherwise

Efficiency requirements for an assignent enumerator

To achieve the maximum efficiency, an assignent enumerator should:

- (Laziness) generate the assignments one-at-a-time.
- (Polynomial Space) require only polynomial space
- (Strong Non-redundancy) be strongly non-redundant
- (Time efficiency) be fast
- [(Symbiosis with $\mathcal{L}$-SOLVE) be able to tale benefit from failure & success information provided by $\mathcal{L}$-SOLVE (e.g., conflict sets, inferred assignments)]

Benefits of (strongly) non-redundant generators

- Non-redundant enumerators avoid generating partial assignments whose unsatisfiability is a propositional consequence of those already generated.
- Strongly non-redundant enumerators avoid generating partial assignments covering areas of the search space which are covered by already-generated ones.
- Strong non-redundancy provides a logical warrant that an already generated assignment will never be generated again. $\Longrightarrow$no extra control required to avoid redundancy.

Efficiency requirements for $\mathcal{L}$-SOLVE()

To achieve the maximum efficiency, $\mathcal{L}$-SOLVE() should:

- (Time efficiency) be fast
- (Polynomial Space) require only polynomial space
- [(Symbiosis with ASSIGN_ENUMERATOR) be able to produce failure & success information (e.g., conflict sets, inferred assignments)]
- [(Incrementality) be incremental: $\mathcal{L}$-SOLVE$(\mu_1 \cup \mu_2)$ reuses computation of $\mathcal{L}$-SOLVE(μ_1)]

Extending existing SAT procedures

General ideas

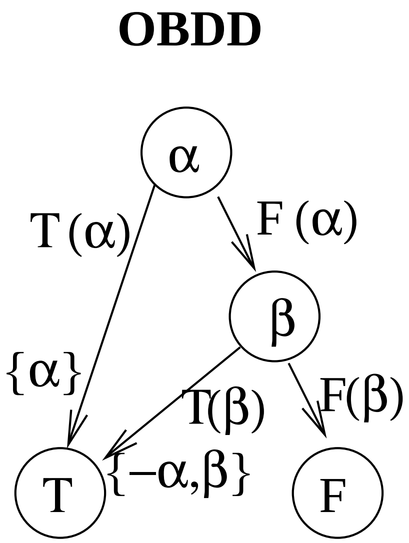
Existing SAT procedures are natural candidates to be used as assignment enumerators.

- Atoms labelled by propositional atoms
- Slight modifications
  (backtrack when assignment found)
- Completeness to be verified!
  (E.g., DPLL with Pure literal)
- Candidates: OBDDs, Semantic Tableaux, DPLL

OBDDs

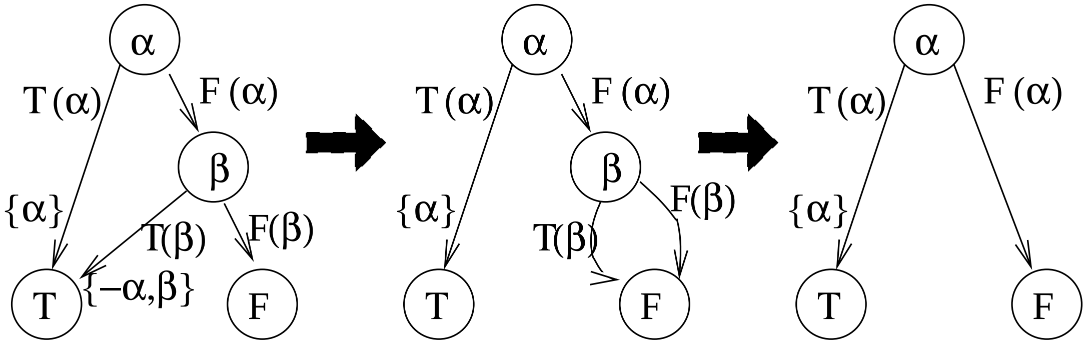
- In an OBDD, the set of paths from the root to (1) represent a complete collection of assignments
- Some may be inconsistent in $\mathcal{L}$
- Reduction: [21, 60, 2]
  1. inconsistent paths from the root to internal nodes are detected
  2. they are redirected to the (0) node
  3. the resulting OBDD is simplified.

OBDD: example



OBDD of $(\alpha \vee \beta \vee \gamma) \wedge (\alpha \vee \beta \vee \neg\gamma)$.

OBDD reduction: example



Reduced OBDD of $(\alpha \vee \beta \vee \gamma) \wedge (\alpha \vee \beta \vee \neg\gamma)$, $\alpha := (x - y \leq 4)$, $\beta := (x - y \leq 2)$.

OBDD: summary

- strongly non-redundant
- time-efficient
- factor sub-graphs
- require exponential memory
- non lazy
- [allow for early pruning]
- [do not allow for backjumping or learning]

Generalized semantic tableaux

- General rules = propositional rules + $\mathcal{L}$-specific rules

$$\left\{ \begin{array}{ccc} \dfrac{\varphi_1 \wedge \varphi_2}{\begin{array}{c}\varphi_1\\ \varphi_2\end{array}} & \dfrac{\neg(\varphi_1 \vee \varphi_2)}{\begin{array}{c}\neg\varphi_1\\ \neg\varphi_2\end{array}} & \dfrac{\neg(\varphi_1 \to \varphi_2)}{\begin{array}{c}\varphi_1\\ \neg\varphi_2\end{array}} \\ & \dfrac{\neg\neg\varphi}{\varphi} & \\ \dfrac{\varphi_1 \vee \varphi_2}{\varphi_1 \quad \varphi_2} & \dfrac{\neg(\varphi_1\wedge\varphi_2)}{\neg\varphi_1 \quad \neg\varphi_2} & \dfrac{\varphi_1\to\varphi_2}{\neg\varphi_1 \quad \varphi_2} \\ \dfrac{\varphi_1 \leftrightarrow \varphi_2}{\begin{array}{cc}\varphi_1 & \neg\varphi_1\\ \varphi_2 & \neg\varphi_2\end{array}} & \dfrac{\neg(\varphi_1\leftrightarrow\varphi_2)}{\begin{array}{cc}\varphi_1 & \neg\varphi_1\\ \neg\varphi_2 & \varphi_2\end{array}} & \end{array} \right\} \cup \left\{ \begin{array}{c} \mathcal{L}\text{-specific} \\ \text{Rules}\end{array} \right\}$$

- Widely used by logicians

Generalized tableau algorithm

```
function $\mathcal{L}$-Tableau($\Gamma$)
    if $A_i \in \Gamma$ and $\neg A_i \in \Gamma$                          /* branch closed */
        then return False;
    if $(\varphi_1 \wedge \varphi_2) \in \Gamma$                             /* ∧-elimination */
        then return $\mathcal{L}$-Tableau($\Gamma \cup \{\varphi_1, \varphi_2\} \setminus \{(\varphi_1 \wedge \varphi_2)\}$);
    if $(\neg\neg\varphi_1) \in \Gamma$                                  /* ¬¬-elimination */
        then return $\mathcal{L}$-Tableau($\Gamma \cup \{\varphi_1\} \setminus \{(\neg\neg\varphi_1)\}$);
    if $(\varphi_1 \vee \varphi_2) \in \Gamma$                             /* ∨-elimination */
        then return     $\mathcal{L}$-Tableau($\Gamma \cup \{\varphi_1\} \setminus \{(\varphi_1 \vee \varphi_2)\}$)  or
                        $\mathcal{L}$-Tableau($\Gamma \cup \{\varphi_2\} \setminus \{(\varphi_1 \vee \varphi_2)\}$);
    ...
    return ($\mathcal{L}$-SOLVE($\Gamma$)= satisfiable);                /* branch expanded */
```

General tableaux: example

Tableau Search Graph

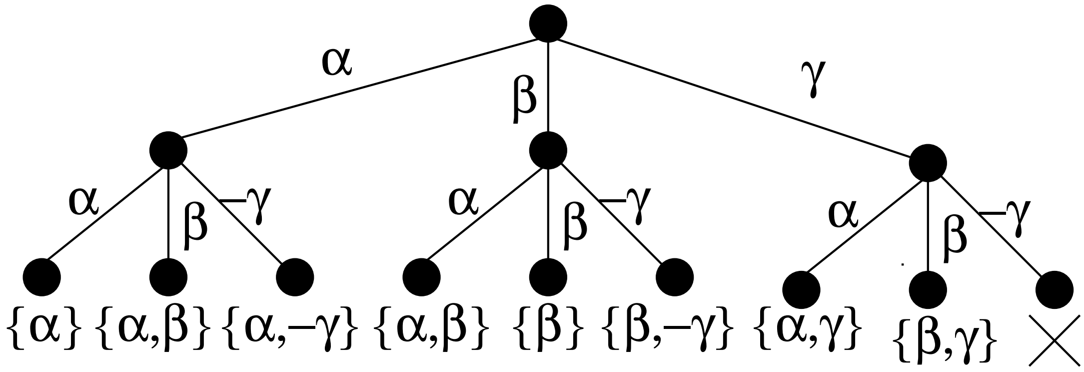


Tableau search graph for $(\alpha \vee \beta \vee \gamma) \wedge (\alpha \vee \beta \vee \neg\gamma)$.

Generalized tableaux: problems

Two main problems [25, 42, 43]

- syntactic branching
  - branch on disjunctions
  - possible many duplicate or subsumed branches $\Longrightarrow$redundant
  - duplicates search (both propositional and domain-dependent)
- no constraint violation detection
  - incapable to detect when current branches violate a constraint $\Longrightarrow$lots of redundant propositional search.

Syntactic branching: example

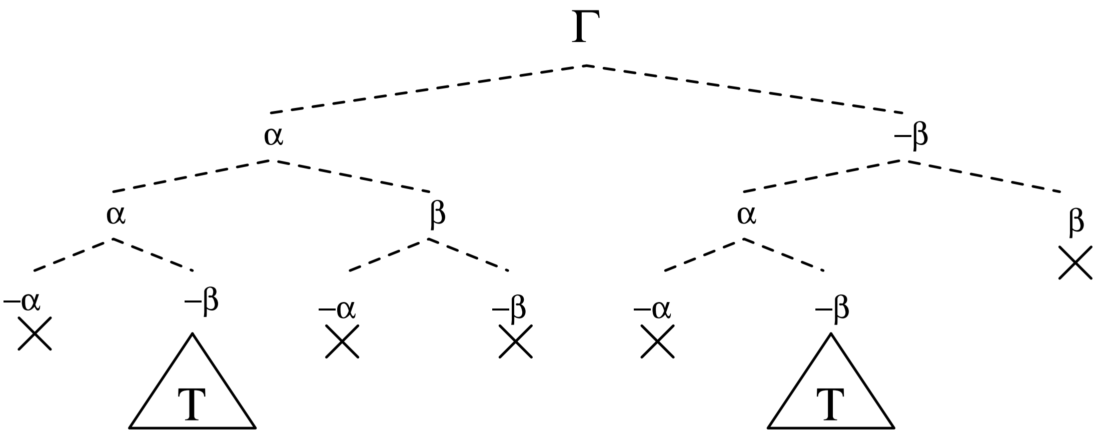


Tableau search graph for $(\alpha \vee \neg\beta) \wedge (\alpha \vee \beta) \wedge (\neg\alpha \vee \neg\beta)$.

Detecting constraints violations: example

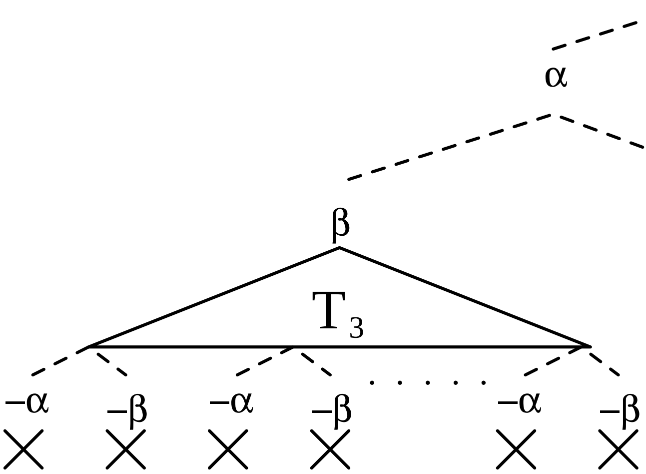


Tableau search graph for $(\alpha \vee \phi_1) \wedge (\beta \vee \phi_2) \wedge \phi_3 \wedge (\neg\alpha \vee \neg\beta)$

Generalized tableaux: summary

– lazy

– require polynomial memory

– redundant

– time-inefficient

– [allow backjumping]

– [do not allow learning]

Tableaux: remark

The word "Tableau" is a bit overloaded in literature. Some existing (and rather efficient) systems, like FacT, DLP [48] and RACER [91], call themselves "Tableau" procedures, although they use a DPLL-like technique to perform boolean reasoning.

"(...) DLP deals with non-determinism in the model construction algorithm by performing a semantic branching search, as in the Davis-Putnam-Logemann-Loveland procedure (DPLL), instead of the syntactic branching search used by most earlier tableaux based implementations (...)" [68]

"(...) The RACER architecture incorporates the following standard optimization techniques: dependency-directed backtracking (...) and DPLL-style semantic branching (...)" [91]

Same for the boolean system KE [25] and its derived systems.

Generalized DPLL

- General rules = propositional rules + $\mathcal{L}$-specific rules

$$\left\{ \begin{array}{ll} \dfrac{\varphi_1 \wedge (l) \wedge \varphi_2}{(\varphi_1 \wedge \varphi_2)[l|\top]} & (Unit) \\ \\ \dfrac{\varphi}{\varphi[l|\top] \quad \varphi[l|\bot]} & (split) \end{array} \right\} \cup \left\{ \begin{array}{c} \mathcal{L}\text{-specific} \\ \text{Rules} \end{array} \right\}$$

- Equivalent formalism described in [90]
- NOTE: No Pure Literal Rule (on non-boolean atoms):
  Pure literal causes incomplete assignment sets!
- if l pure in φ, typically $\varphi[l|\top]$ is investigated before $\varphi[l|\bot]$

Pure literal and Generalized DPLL: Example

$$\begin{aligned}\varphi = \; & ((x - y \le 1) \vee A_1) \wedge \\ & ((y - z \le 2) \vee A_2) \wedge \\ & (\neg(x - z \le 4) \vee A_2) \wedge \\ & (\neg A_2 \vee A_3) \wedge \\ & (\neg A_2 \vee \neg A_3)\end{aligned}$$

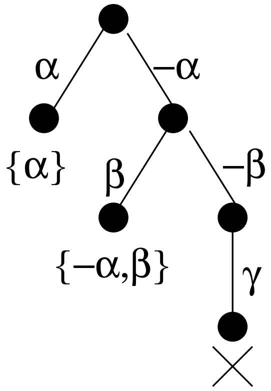
- A satisfiable assignment propositionally satisfying φ is: $\mu = \{A_1, \neg A_2, (y - z \le 2), \neg(x - z \le 4)\}$
- No satisfiable assignment propositionally satisfying φ contains $(x - y \le 1)$
- Pure literal may assign $(x - y \le 1) := \top$ as first step $\Longrightarrow$return unsatisfiable.

Generalized DPLL algorithm

```
function $\mathcal{L}$-DPLL($\varphi, \mu$)
    if $\varphi = \top$                                          /* base      */
        then return ($\mathcal{L}$-SOLVE($\mu$)=satisfiable);
    if $\varphi = \bot$                                          /* backtrack */
        then return False;
    if {a unit clause $(l)$ occurs in $\varphi$}                 /* unit      */
        then return $\mathcal{L}$-DPLL(assign$(l, \varphi), \mu \wedge l$);
    l := choose-literal($\varphi$);                              /* split     */
    return  $\mathcal{L}$-DPLL(assign$(l, \varphi), \mu \wedge l$)  or
            $\mathcal{L}$-DPLL(assign$(\neg l, \varphi), \mu \wedge \neg l$);
```

General DPLL: example

DPLL search graph



DPLL search graph for $(\alpha \vee \beta \vee \gamma) \wedge (\alpha \vee \beta \vee \neg\gamma)$.

Generalized DPLL vs. generalized tableaux

Two big advantages: [25, 42, 43]

- semantic vs. syntactic branching
  - branch on truth values
  - no duplicate or subsumed branches $\Longrightarrow$strongly non redundant
  - no search duplicates
- constraint violation detection
  - backtracks as soon as the current branch violates a constraint $\Longrightarrow$no redundant propositional search.

Semantic branching: example

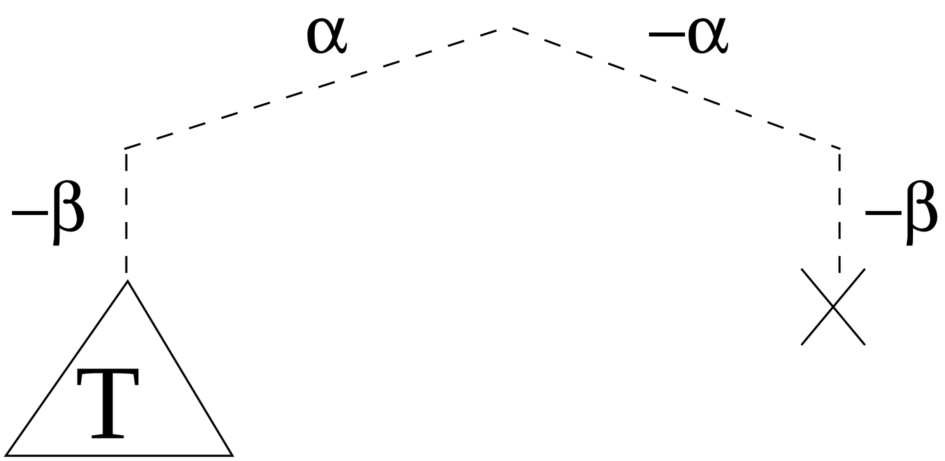
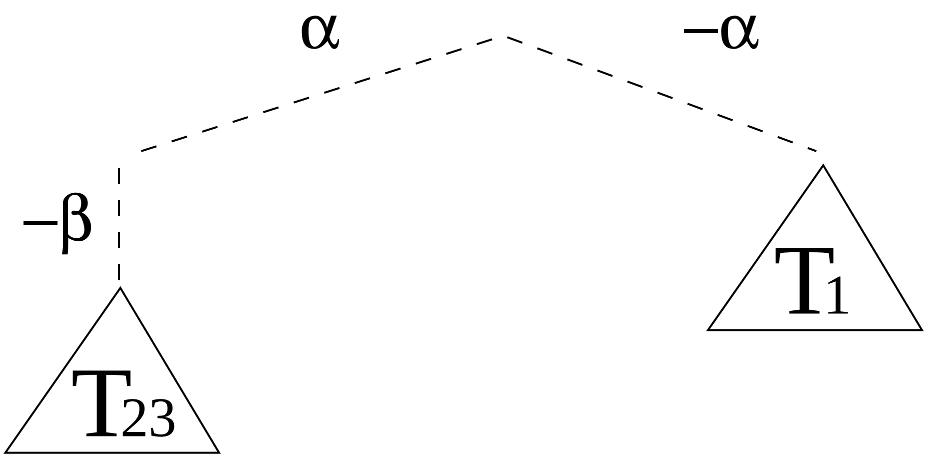


Tableau search graph for $(\alpha \vee \neg\beta) \wedge (\alpha \vee \beta) \wedge (\neg\alpha \vee \neg\beta)$.

Detecting constraints violations: example



DPLL search graph for $(\alpha \vee \phi_1) \wedge (\beta \vee \phi_2) \wedge \phi_3 \wedge (\neg\alpha \vee \neg\beta)$

Generalized DPLL vs. generalized tableaux: remarks

- ▷ Generalized tableaux reason on subformula instances
- ▷ Generalized DPLL reasons on atoms
  $\Longrightarrow$all instances of an atom are handled contemporarily
- ▷ If the atoms have no multiple occurrences, the benefits of DPLL vs. tableaux are negligible (unless learning is used)

Generalized DPLL: summary

- lazy
- require polynomial memory
- strongly non redundant
- time-efficient
- [allow backjumping and learning]

Optimizations

Possible Improvements

- Preprocessing atoms [41, 48, 7]
- Static learning [3]
- Early pruning [41, 21, 6]
- Enhanced Early pruning [6]
- Backjumping [48, 95]
- Memoizing [48, 37]
- Learning [48, 95]
- Forward Checking [3]
- Triggering [95, 6]
- ...

Preprocessing atoms [41, 48, 7]

Source of inefficiency: semantically equivalent but syntactically different atoms are not recognized to be identical [resp. one the negation of the other] $\Longrightarrow$they may be assigned different [resp. identical] truth values.

Solution: rewrite trivially equivalent atoms into one.

Preprocessing atoms (cont.)

- Sorting: $(v_1 + v_2 \leq v_3 + 1)$, $(v_2 + v_1 \leq v_3 + 1)$, $(v_1 + v_2 - 1 \leq v_3) \Longrightarrow (v_1 + v_2 - v_3 \leq 1))$;
- Rewriting dual operators: $(v_1 < v_2)$, $(v_1 \geq v_2) \Longrightarrow (v_1 < v_2)$, $\neg(v_1 < v_2)$
- Exploiting associativity: $(v_1 + (v_2 + v_3) = 1)$, $((v_1 + v_2) + v_3) = 1) \Longrightarrow (v_1 + v_2 + v_3 = 1)$;
- Factoring $(v_1 + 2.0v_2 \leq 4.0)$, $(-2.0v_1 - 4.0v_2 \geq -8.0)$, $\Longrightarrow (0.25v_1 + 0.5v_2 \leq 1.0)$;
- Exploiting properties of $\mathcal{L}$: $(v_1 \leq 3)$, $(v_1 < 4) \Longrightarrow (v_1 \leq 3)$ if $v_1 \in \mathbb{Z}$;
- ...

Preprocessing atoms: summary

- Very efficient with DPLL
- Presumably very efficient with OBDDs
- Scarcely efficient with semantic tableaux

Static learning [3]

- Rationale: Many literals are mutually exclusive (e.g., $(x - y < 3), \neg(x - y < 5)$)
- Preprocessing step: detect these literals and add binary clauses to the input formula: (e.g., $\neg(x - y < 3) \vee (x - y < 5)$)
- (with DPLL) assignments including both literals are never generated.
- requires $O(|\varphi|^2)$ steps.

Static learning (cont.)

- Very efficient with DPLL
- Possibly very efficient with OBDDs (?)
- Completely ineffective with semantic tableaux

Early pruning [41, 21, 6]

- rationale: if an assignment μ' is unsatisfiable, then all its extensions are unsatisfiable.
- the unsatisfiability of μ' detected during its construction, $\Longrightarrow$avoids checking the satisfiability of all the up to $2^{|Atoms(\varphi)|-|\mu'|}$ assignments extending μ'.
- Introduce a satisfiability test on incomplete assignments just before every branching step:

```
if Likely-Unsatisfiable($\mu$)            /* early pruning */
    if ($\mathcal{L}$-SOLVE($\mu$) = $False$)
        then return False;
```

DPLL+Early pruning

```
function $\mathcal{L}$-DPLL($\varphi, \mu$)
    if $\varphi = \top$                                              /* base          */
        then return ($\mathcal{L}$-SOLVE($\mu$)=satisfiable);
    if $\varphi = \bot$                                              /* backtrack */
        then return False;
    if {a unit clause $(l)$ occurs in $\varphi$}                     /* unit          */
        then return $\mathcal{L}$-DPLL($assign(l, \varphi), \mu \wedge l$);
    if Likely-Unsatisfiable($\mu$)                                   /* early pruning */
        if ($\mathcal{L}$-SOLVE($\mu$) = False)
            then return False;
    l := choose-literal($\varphi$);                                  /* split         */
    return  $\mathcal{L}$-DPLL($assign(l, \varphi), \mu \wedge l$)  or
            $\mathcal{L}$-DPLL($assign(\neg l, \varphi), \mu \wedge \neg l$);
```

Early pruning: example

$$
\begin{aligned}
\varphi = \quad & \{\neg(2v_2 - v_3 > 2) \vee A_1\} \wedge \\
& \{\neg A_2 \vee (2v_1 - 4v_5 > 3)\} \wedge \\
& \{(3v_1 - 2v_2 \leq 3) \vee A_2\} \wedge \\
& \{\neg(2v_3 + v_4 \geq 5) \vee \neg(3v_1 - v_3 \leq 6) \vee \neg A_1\} \wedge \\
& \{A_1 \vee (3v_1 - 2v_2 \leq 3)\} \wedge \\
& \{(v_1 - v_5 \leq 1) \vee (v_5 = 5 - 3v_4) \vee \neg A_1\} \wedge \\
& \{A_1 \vee (v_3 = 3v_5 + 4) \vee A_2\}.
\end{aligned}
$$

– Suppose it is built the intermediate assignment:

$$\mu' = \neg(2v_2 - v_3 > 2) \wedge \neg A_2 \wedge (3v_1 - 2v_2 \leq 3) \wedge \neg(3v_1 - v_3 \leq 6).$$

– If $\mathcal{L}$-SOLVE is invoked on μ', it returns *False*, and $\mathcal{L}$-DPLL backtracks without exploring any extension of μ'.

Early pruning: drawback

- Reduces drastically the search
- Drawback: possibly lots of useless calls to $\mathcal{L}$-SOLVE
  $\Longrightarrow$to be used with care when $\mathcal{L}$-SOLVE calls recursively $\mathcal{L}$-SAT (e.g., with modal logics)
- Roughly speaking, worth doing when each branch saves at least one branching
- Possible solutions:
  - introduce a selective heuristic Likely-unsatisfiable
  - use incremental versions of $\mathcal{L}$-SOLVE

  one split.

Early pruning: Likely-unsatisfiable

- Rationale: if no literal which may likely cause conflict with the previous assignment has been added since last call, return false.
- Examples: return false if they are added only
  - boolean literals
  - disequalities $(x - y \neq 3)$
  - atoms introducing new variables $(x - z \neq 3)$
  - ...

Early pruning: incrementality of $\mathcal{L}$-SOLVE

- With early pruning, lots of incremental calls to $\mathcal{L}$-SOLVE:

$$\begin{array}{lll} \mathcal{L}\text{-SOLVE}(\mu) & \Longrightarrow & \text{satisfiable} \\ \mathcal{L}\text{-SOLVE}(\mu \cup \mu') & \Longrightarrow & \text{satisfiable} \\ \mathcal{L}\text{-SOLVE}(\mu \cup \mu' \cup \mu'') & \Longrightarrow & \text{satisfiable} \\ \ldots & & \end{array}$$

- $\mathcal{L}$-SOLVE incremental: $\mathcal{L}$-SOLVE$(\mu_1 \cup \mu_2)$ reuses computation of $\mathcal{L}$-SOLVE(μ_1) without restarting from scratch $\Longrightarrow$ lots of computation saved
- requires saving the status of $\mathcal{L}$-SOLVE

Early pruning: summary

- Very efficient with DPLL & OBDDs
- Possibly very efficient with semantic tableaux (?)
- In some cases may introduce big overhead (e.g., modal logics)
- Benefits if $\mathcal{L}$-SOLVE is incremental

Enhanced Early Pruning [6]

- In early pruning, $\mathcal{L}$-SOLVE is not effective if it returns "satisfiable".
- $\mathcal{L}$-SOLVE(μ) may be able to derive (easily) a sub-assignment η s.t. $\mu \models \eta$, and return it.
- The literals in η are then unit-propagated away.

Enhanced Early Pruning: Examples

(We assume that all the following literals occur in φ.)

- If $(v_1 - v_2 \leq 4) \in \mu$ and $(v_1 - v_2 \leq 6) \notin \mu$, then $\mathcal{L}$-SOLVE can derive $(v_1 - v_2 \leq 6)$ from μ.
- If $(v_1 - v_3 > 2), (v_2 = v_3) \in \mu$ and $(v_1 - v_2 > 2) \notin \mu$, then $\mathcal{L}$-SOLVEcan derive $(v_1 - v_2 > 2)$ from μ.

Enhanced Early Pruning: summary

- Further improves efficiency with DPLL
- Presumably scarcely effective with semantic tableaux
- Effective with OBDDs?
- Requires a sophisticated $\mathcal{L}$-SOLVE (able to perform deductive inference)

Backjumping (driven by $\mathcal{L}$-SOLVE) [48, 95]

- Similar to SAT backjumping
- Rationale: same as for early pruning
- Idea: when a branch is found unsatisfiable in $\mathcal{L}$,
  1. $\mathcal{L}$-SOLVE returns the conflict set causing the failure
  2. $\mathcal{L}$-SAT backtracks to the most recent branching point in the conflict set

Backjumping: Example

$$\begin{aligned}\varphi = \quad & \{\neg(2v_2 - v_3 > 2) \vee A_1\} \wedge \\ & \{\neg A_2 \vee (2v_1 - 4v_5 > 3)\} \wedge \\ & \{(3v_1 - 2v_2 \leq 3) \vee A_2\} \wedge \\ & \{\neg(2v_3 + v_4 \geq 5) \vee \neg(3v_1 - v_3 \leq 6) \vee \neg A_1\} \wedge \\ & \{A_1 \vee (3v_1 - 2v_2 \leq 3)\} \wedge \\ & \{(v_1 - v_5 \leq 1) \vee (v_5 = 5 - 3v_4) \vee \neg A_1\} \wedge \\ & \{A_1 \vee (v_3 = 3v_5 + 4) \vee A_2\}.\end{aligned}$$

$$\mu = \{\neg(2v_2 - v_3 > 2), \neg A_2, (3v_1 - 2v_2 \leq 3), (v_1 - v_5 \leq 1), \neg(3v_1 - v_3 \leq 6), (v_3 = 3v_5 + 4)\}.$$

- $\mathcal{L}\text{-SOLVE}(\mu)$ returns *false* with the conflict set:

$$\{(3v_1 - 2v_2 \leq 3), \neg(2v_2 - v_3 > 2), \neg(3v_1 - v_3 \leq 6)\}$$

- $\mathcal{L}\text{-SAT}$ can jump back directly to the branching point $\neg(3v_1 - v_3 \leq 6)$, without branching on $(v_3 = 3v_5 + 4)$.

Backjumping vs. Early Pruning

- Backjumping requires no extra calls to $\mathcal{L}\text{-SOLVE}$
- Effectiveness depends on the conflict set C, i.e., on how recent the most recent branching point in C is.
- Example: no pruning effect with the conflict set:

$$\{(v_1 - v_5 \leq 1), (v_3 = 3v_5 + 4), \neg(3v_1 - v_3 \leq 6)\}$$

- Same pruning effect as with Early Pruning only with the best conflict set
- More effective than Early Pruning only when the overhead compensates the pruning effect (e.g., modal logics with high depths).

Backjumping: summary

- Very efficient with DPLL
- Never applied to OBDDs
- Very efficient with semantic tableaux
- Alternative to but less effective than early pruning.
- No significant overhead
- $\mathcal{L}$-SOLVE must be able to detect conflict sets.

Memoizing [48, 37]

- Idea 1:
  - When a conflict set C is revealed, then C can be cached into an ad hoc data structure
  - $\mathcal{L}$-SOLVE(μ) checks first if (any subset of) μ is cached. If yes, returns unsatisfiable.
- Idea 2:
  - When a satisfying (sub)-assignment μ' is found, then μ' can be cached into an ad hoc data structure
  - $\mathcal{L}$-SOLVE(μ) checks first if (any superset of) μ is cached. If yes, returns satisfiable.

Memoizing (cont.)

- Can dramatically prune search.
- May cause a blowup in memory.
- Applicable also to semantic tableaux.
- Idea 1 subsumed by learning.

Learning (driven by $\mathcal{L}$-SOLVE) [48, 95]

- Similar to SAT learning
- Idea: When a conflict set C is revealed, then $\neg C$ can be added to the clause set
  $\Longrightarrow$DPLL will never again generate an assignment containing C.
- May avoid a lot of redundant search.
- Problem: may cause a blowup in space
  $\Longrightarrow$techniques to control learning and to drop learned clauses when necessary

Learning: example

- $\mathcal{L}$-SOLVE returns the conflict set:
  $\{(3v_1 - 2v_2 \le 3), \neg(2v_2 - v_3 > 2), \neg(3v_1 - v_3 \le 6)\}$
- it is added the clause
  $\neg(3v_1 - 2v_2 \le 3) \vee (2v_2 - v_3 > 2) \vee (3v_1 - v_3 \le 6)$
- Prunes up to 2^{N-3} assignments
  $\Longrightarrow$the smaller the conflict set, the better.

Learning: summary

- Very efficient with DPLL
- Never applied to OBDDs
- Completely ineffective with semantic tableaux
- May cause memory blowup
- $\mathcal{L}$-SOLVE must be able to detect conflict sets.

Forward Checking [3]

- Idea: if $\mu \wedge l \wedge l'$ inconsistent, then $\mu \wedge l \models \neg l'$
- $assign(\varphi, l)$ substituted with $fc\_assign(\varphi, \mu \wedge l)$: $fc\_assign(\varphi, \mu \wedge l)$ replaces $cl \vee l'$ with cl if $\mathcal{L}\text{-SOLVE}(\mu \wedge l \wedge l')$ returns false, for every l'
- can significantly prune search
- significant overhead: many possibly redundant calls to $\mathcal{L}$-SOLVE

Triggering [95, 6]

Proposition Let C be a non-boolean atom occurring only positively [resp. negatively] in φ. Let $\mathcal{M}$ be a complete set of assignments for φ, and let

$$\mathcal{M}' := \{\mu_j/\neg C\} | \mu_j \in \mathcal{M}\} \quad [resp.\ \{\mu_j/C\} | \mu_j \in \mathcal{M}\}].$$

Then φ is satisfiable if and only if there exist a satisfiable $\eta' \in \mathcal{M}'$ s.t. $\eta' \models_p \varphi$.

Proof (Sketch) The "if" case is trivial. If φ is satisfiable, then there is a satisfiable $\eta \in \mathcal{M}$ s.t. $\eta \models_p \varphi$ because $\mathcal{M}$ is complete. If $\neg C \notin \eta$, then the thesis holds with $\eta' := \eta$. If $\neg C \in \eta$, then let $\eta' := \eta/\neg C$. η' is trivially satisfiable. As $\eta \models_p \varphi$ and C occurs only positively in φ, $\eta' \models_p \varphi$.

Triggering (cont.)

- If we have non-boolean atoms occurring only positively [negatively] in φ, we can drop any negative [positive] occurrence of them from the assignment to be checked by $\mathcal{L}$-SOLVE
- Particularly useful when we deal with equality atoms (e.g., $(v_1 - v_2 = 3.2)$), as handling negative equalities like $(v_1 - v_2 \neq 3.2)$ forces splitting: $(v_1 - v_2 > 3.2) \vee (v_1 - v_2 < 3.2)$.

Application Fields

- Modal Logics [41, 48, 43, 37]
- Description Logics [42, 48]
- Boolean+Mathematical reasoning (Temporal reasoning [3], Resource Planning [95], Verification of Timed Systems [60, 6, 9, 84, 29, 65, 70], Verification of systems with arithmetical operators [21, 89], verification of hybrid systems [8]
- decision procedures in combined theories [62, 63, 81, 31, 7, 6, 12, 13, 89, 29, 56, 78, 90, 87, 86]
- ...

Case study: Modal Logic(s)

Satisfiability in Modal logics

- Propositional logics enhanced with modal operators $\Box_i$, K_i, etc.
- Used to represent complex concepts like knowledge, necessity/possibility, etc.
- Based on Kripke's possible worlds semantics [54]
- Very hard to decide [45, 44] (typically PSPACE-complete or worse)
- Strictly related to Description Logics [72] (ex: $K(m) \Longleftrightarrow \mathcal{ALC}$)
- Various fields of application: AI, formal verification, knowledge bases, etc.

Syntax

Given a non-empty set of primitive propositions $\mathcal{A} = \{A_1, A_2, \ldots\}$ and a set of m modal operators $\mathcal{B} = \{\Box_1, \ldots, \Box_m\}$, the modal language $\mathcal{L}$ is the least set of formulas containing $\mathcal{A}$, closed under the set of propositional connectives $\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$ and the set of modal operators in $\mathcal{B}$.

- depth(φ) is the maximum number of nested modal operators in φ.
- "$\Box_i\varphi$" can be interpreted as "Agent i knows φ"

Semantics

- A Kripke structure for $\mathcal{L}$ is a tuple $M = < \mathcal{U}, \pi, \mathcal{R}_1, \ldots, \mathcal{R}_m >$, where
  - $\mathcal{U}$ is a set of states $u_1, u_2, \ldots$
  - π is a function $\pi : \mathcal{A} \times \mathcal{U} \longmapsto \{\top, \bot\}$,
  - each $\mathcal{R}_r$ is a binary relation on the states of $\mathcal{U}$.

Semantics (cont)

Given M, u s.t. $u \in \mathcal{U}$, $M, u \models \varphi$ is defined as follows:

$$
\begin{array}{lcl}
M, u \models A_i,\ A_i \in \mathcal{A} & \Longleftrightarrow & \pi(A_i, u) = \top; \\
M, u \models \neg\varphi_1 & \Longleftrightarrow & M, u \not\models \varphi_1; \\
M, u \models \varphi_1 \wedge \varphi_2 & \Longleftrightarrow & M, u \models \varphi_1 \textit{ and } M, u \models \varphi_2; \\
M, u \models \varphi_1 \vee \varphi_2 & \Longleftrightarrow & M, u \models \varphi_1 \textit{ or } M, u \models \varphi_2. \\
\ldots & & \\
M, u \models \Box_r\varphi_1,\ \Box_r \in \mathcal{B} & \Longleftrightarrow & M, v \models \varphi_1 \text{ for every } v \in \mathcal{U} \\
 & & \text{s.t. } \mathcal{R}_r(u, v) \text{ holds in } M. \\
M, u \models \neg\Box_r\varphi_1,\ \Box_r \in \mathcal{B} & \Longleftrightarrow & M, v \models \neg\varphi_1 \text{ for some } v \in \mathcal{U} \\
 & & \text{s.t. } \mathcal{R}_r(u, v) \text{ holds in } M.
\end{array}
$$

Semantics (cont)

The (normal) modal logics vary with the properties of $\mathcal{R}_r$:

Axiom	Property of $\mathcal{R}$	Description
B	symmetric	$\forall\, u\, v\ \mathcal{R}(u,v) \Longrightarrow \mathcal{R}(v,u)$
D	serial	$\forall\, u\, \exists\, v\ \mathcal{R}(u,v)$
T	reflexive	$\forall\, u\ \mathcal{R}(u,u)$
4	transitive	$\forall\, u\, v\, w\ \mathcal{R}(u,v)\ e\ \mathcal{R}(v,w) \Longrightarrow \mathcal{R}(u,w)$
5	euclidean	$\forall\, u\, v\, w\ \mathcal{R}(u,v)\ e\ \mathcal{R}(u,w) \Longrightarrow \mathcal{R}(v,w)$

Normal Modal Logic	Properties of $\mathcal{R}_r$
K	—
KB	symmetric
KD	serial
KT = KDT (T)	reflexive
K4	transitive
K5	euclidean
KBD	symmetric and serial
KBT = KBDT (B)	symmetric and reflexive
KB4 = KB5 = KB45	symmetric and transitive
KD4	serial and transitive
KD5	serial and euclidean
KT4 = KDT4 (S4)	reflexive and transitive
KT5 = KBD4 = KBD5 = KBT4 = KBT5 = KDT5 = KT45 = KBD45 = KBT45 = KDT45 = KBDT4 = KBDT5 = KBDT45 (S5)	reflexive, transitive and symmetric (equivalence)
K45	transitive and euclidean
KD45	serial, transitive and euclidean

Axiomatic framework

- Basic Axioms:

$I.\quad \alpha \to (\beta \to \alpha),$

$II.\quad (\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma)),$

$III.\quad (\neg\alpha \to \beta) \to ((\neg\alpha \to \neg\beta) \to \alpha),$

$K:\quad \Box_r\alpha \to (\Box_r(\alpha \to \beta) \to \Box_r\beta)$

- Specific Axioms:

$B.\quad \alpha \to \Box_r\neg\Box_r\neg\alpha,$

$D.\quad \Box_r\alpha \to \neg\Box_r\neg\alpha,$

$T.\quad \Box_r\alpha \to \alpha,$

$4.\quad \Box_r\alpha \to \Box_r\Box_r\alpha,$

$5.\quad \neg\Box_r\alpha \to \Box_r\neg\Box_r\alpha.$

Axiomatic framework (cont.)

- Inference rules:

$$\frac{\alpha \quad \alpha \to \beta}{\beta}\ \text{(modus ponens)},$$

$$\frac{\alpha}{\Box_r \alpha}\ \text{(necessitation)}.$$

- Correctness & completeness:

  φ is valid $\Longleftrightarrow$ φ can be deduced

Tableaux for modal K(m)/$\mathcal{ALC}$ [32]

- Rules = tableau rules + $K(m)$-specific rules

$$\left\{ \begin{array}{ccc}
\dfrac{\varphi_1 \wedge \varphi_2}{\begin{array}{c}\varphi_1\\ \varphi_2\end{array}} & \dfrac{\neg(\varphi_1 \vee \varphi_2)}{\begin{array}{c}\neg\varphi_1\\ \neg\varphi_2\end{array}} & \dfrac{\neg(\varphi_1 \to \varphi_2)}{\begin{array}{c}\varphi_1\\ \neg\varphi_2\end{array}} \\
 & \dfrac{\neg\neg\varphi}{\varphi} & \\
\dfrac{\varphi_1 \vee \varphi_2}{\varphi_1 \quad \varphi_2} & \dfrac{\neg(\varphi_1 \wedge \varphi_2)}{\neg\varphi_1 \quad \neg\varphi_2} & \dfrac{\varphi_1 \to \varphi_2}{\neg\varphi_1 \quad \varphi_2} \\
\dfrac{\varphi_1 \leftrightarrow \varphi_2}{\begin{array}{cc}\varphi_1 & \neg\varphi_1\\ \varphi_2 & \neg\varphi_2\end{array}} & \dfrac{\neg(\varphi_1 \leftrightarrow \varphi_2)}{\begin{array}{cc}\varphi_1 & \neg\varphi_1\\ \neg\varphi_2 & \varphi_2\end{array}} &
\end{array} \right\} \cup \left\{ \dfrac{\Box_r\alpha_1, \ldots, \Box_r\alpha_N, \neg\Box_r\beta_j}{\alpha_1, \ldots, \alpha_N, \neg\beta_j} \right\}$$

DPLL for K(m)/$\mathcal{ALC}$: K-SAT [41, 42]

– Rules = DPLL rules + $K(m)$-specific rules

$$\left\{ \begin{array}{l} \dfrac{\varphi_1 \wedge (l) \wedge \varphi_2}{(\varphi_1 \wedge \varphi_2)[l|\top]} \ (Unit) \\ \\ \dfrac{\varphi}{\varphi[l|\top] \quad \varphi[l|\bot]} \ (split) \end{array} \right\} \cup \left\{ \dfrac{\Box_r\alpha_1,\ldots,\Box_r\alpha_N,\neg\Box_r\beta_j}{\alpha_1,\ldots,\alpha_N,\neg\beta_j} \right\}$$

The K-SAT algorithm [41, 42]

```
function K-SAT($\varphi$)
    return K-DPLL($\varphi, \top$);

function K-DPLL($\varphi, \mu$)
    if $\varphi = \top$                                  /* base        */
        then return K-SOLVE($\mu$);
    if $\varphi = \bot$                                  /* backtrack */
        then return False;
    if {a unit clause $(l)$ occurs in $\varphi$}          /* unit        */
        then return K-DPLL($assign(l, \varphi), \mu \wedge l$);
    if Likely-Unsatisfiable($\mu$)                       /* early pruning */
        if not K-SOLVE($\mu$)
            then return False;
    $l$ := choose-literal($\varphi$);                    /* split       */
    return  K-DPLL($assign(l, \varphi), \mu \wedge l$) or
            K-DPLL($assign(\neg l, \varphi), \mu \wedge \neg l$);
```

The K-SAT algorithm (cont.)

```
function K-SOLVE($\bigwedge_i \Box_1 \alpha_{1i} \wedge \bigwedge_j \neg\Box_1 \beta_{1j} \wedge \ldots \wedge \bigwedge_i \Box_m \alpha_{mi} \wedge \bigwedge_j \neg\Box_m \beta_{mj} \wedge \gamma$)
    for each box index $r$ do
        if not K-SOLVE$_{restr}$($\bigwedge_i \Box_r \alpha_{ri} \wedge \bigwedge_j \neg\Box_r \beta_{rj}$)
            then return False;
    return True;

function K-SOLVE$_{restr}$($\bigwedge_i \Box_r \alpha_{ri} \wedge \bigwedge_j \neg\Box_r \beta_{rj}$)
    for each conjunct "$\neg\Box_r \beta_{rj}$" do
        if not K-SAT($\bigwedge_i \alpha_{ri} \wedge \neg\beta_{rj}$)
            then return False;
    return True;
```

K-SAT: Example

$$
\begin{array}{rl}
\varphi = & \{\neg\Box_1(\neg A_3 \vee \neg A_1 \vee A_2) \vee A_1 \vee A_5\} \wedge \\
& \{\neg A_2 \vee \neg A_5 \vee \Box_2(\neg A_2 \vee \neg A_4 \vee \neg A_3)\} \wedge \\
& \{A_1 \vee \Box_2(\neg A_4 \vee A_5 \vee A_2) \vee A_2\} \wedge \\
& \{\neg\Box_2(A_4 \vee \neg A_3 \vee A_1) \vee \neg\Box_1(A_4 \vee \neg A_2 \vee A_3) \vee \neg A_5\} \wedge \\
& \{\neg A_3 \vee A_1 \vee \Box_2(\neg A_4 \vee A_5 \vee A_2)\} \wedge \\
& \{\Box_1(\neg A_5 \vee A_4 \vee A_3) \vee \Box_1(\neg A_1 \vee A_4 \vee A_3) \vee \neg A_1\} \wedge \\
& \{A_1 \vee \Box_1(\neg A_2 \vee A_1 \vee A_4) \vee A_2\}
\end{array}
$$

$\Downarrow$ K-SOLVE()

$$
\begin{array}{rllll}
\mu = & \Box_1(\neg A_5 \vee A_4 \vee A_3) \wedge & \Box_1(\neg A_2 \vee A_1 \vee A_4) \wedge & & [\bigwedge_i \Box_1 \alpha_{1i}] \\
& \neg\Box_1(\neg A_3 \vee \neg A_1 \vee A_2) \wedge & \neg\Box_1(A_4 \vee \neg A_2 \vee A_3) \wedge & & [\bigwedge_j \neg\Box_1 \beta_{1j}] \\
& \Box_2(\neg A_4 \vee A_5 \vee A_2) \wedge & & & [\bigwedge_i \Box_2 \alpha_{2i}] \\
& \neg A_2. & & & [\gamma]
\end{array}
$$

K-SAT: Example (cont.)

$$\begin{array}{lllll}
\mu & = & \Box_1(\neg A_5 \vee A_4 \vee A_3) \wedge & \Box_1(\neg A_2 \vee A_1 \vee A_4) \wedge & [\bigwedge_i \Box_1 \alpha_{1i}] \\
 & & \neg\Box_1(\neg A_3 \vee \neg A_1 \vee A_2) \wedge & \neg\Box_1(A_4 \vee \neg A_2 \vee A_3) \wedge & [\bigwedge_j \neg\Box_1 \beta_{1j}] \\
 & & \Box_2(\neg A_4 \vee A_5 \vee A_2) \wedge & & [\bigwedge_i \Box_2 \alpha_{2i}] \\
 & & \neg A_2. & & [\gamma]
\end{array}$$

$$\Downarrow \ \text{K-SOLVE}_{restr}()$$

$$\begin{array}{lllll}
\mu^1 & = & \Box_1(\neg A_5 \vee A_4 \vee A_3) \wedge & \Box_1(\neg A_2 \vee A_1 \vee A_4) \wedge & [\bigwedge_i \Box_1 \alpha_{1i}] \\
 & & \neg\Box_1(\neg A_3 \vee \neg A_1 \vee A_2) \wedge & \neg\Box_1(A_4 \vee \neg A_2 \vee A_3) & [\bigwedge_j \neg\Box_1 \beta_{1j}] \\
\mu^2 & = & \Box_2(\neg A_4 \vee A_5 \vee A_2) & & [\bigwedge_i \Box_2 \alpha_{2i}].
\end{array}$$

$$\Downarrow \ \text{K-SAT()}$$

$$\begin{array}{lll}
\varphi^{11} & = & (\neg A_5 \vee A_4 \vee A_3) \wedge (\neg A_2 \vee A_1 \vee A_4) \wedge A_3 \wedge A_1 \wedge \neg A_2, \\
\varphi^{12} & = & (\neg A_5 \vee A_4 \vee A_3) \wedge (\neg A_2 \vee A_1 \vee A_4) \wedge \neg A_4 \wedge A_2 \wedge \neg A_3
\end{array}$$

K-SAT: Example (cont.)

$$\begin{array}{lll}
\varphi^{11} & = & (\neg A_5 \vee A_4 \vee A_3) \wedge (\neg A_2 \vee A_1 \vee A_4) \wedge A_3 \wedge A_1 \wedge \neg A_2, \\
\varphi^{12} & = & (\neg A_5 \vee A_4 \vee A_3) \wedge (\neg A_2 \vee A_1 \vee A_4) \wedge \neg A_4 \wedge A_2 \wedge \neg A_3
\end{array}$$

$$\Downarrow \ \text{K-SOLVE()}$$

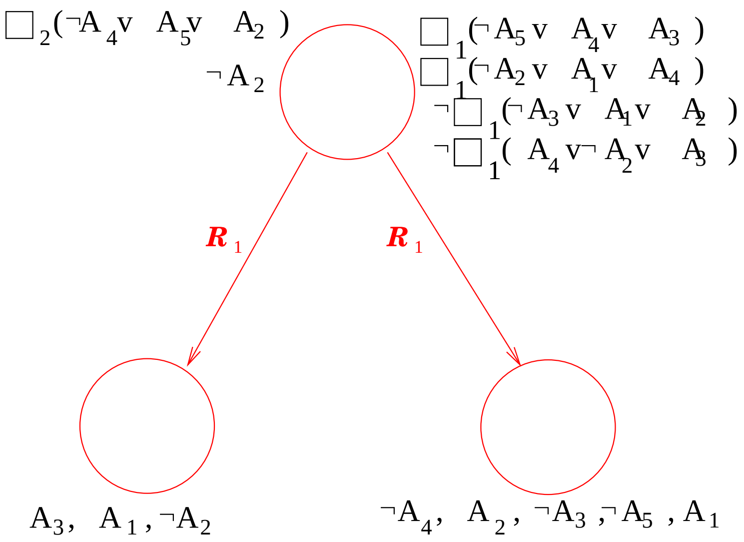
$$\begin{array}{lll}
\mu^{11} & = & A_3 \wedge A_1 \wedge \neg A_2 \\
\mu^{12} & = & \neg A_4 \wedge A_2 \wedge \neg A_3 \wedge \neg A_5 \wedge A_1
\end{array}$$

$$\Downarrow$$

$$\mathit{Satisfiable}$$

Example

Resulting Kripke Model:



$\Box_2(\neg A_4 \vee A_5 \vee A_2)$

$\neg A_2$

$\Box_1(\neg A_5 \vee A_4 \vee A_3)$

$\Box_1(\neg A_2 \vee A_1 \vee A_4)$

$\neg\Box_1(\neg A_3 \vee A_1 \vee A_2)$

$\neg\Box_1(A_4 \vee \neg A_2 \vee A_3)$

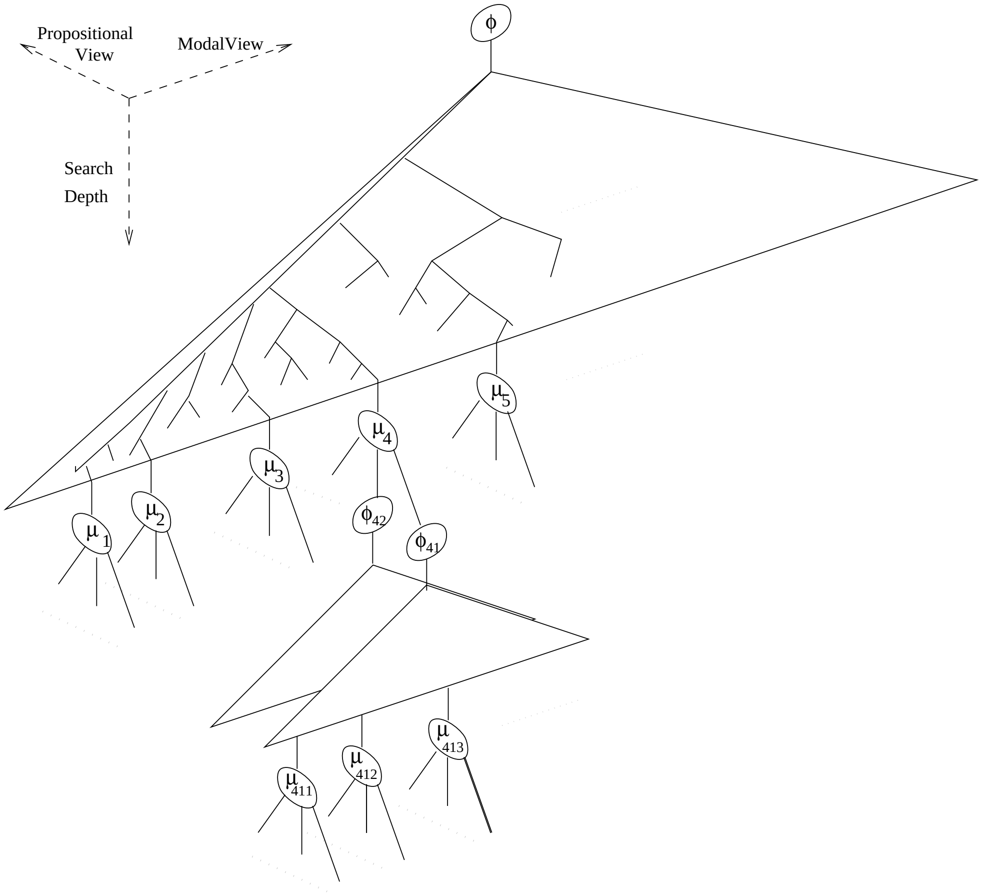
$A_3, A_1, \neg A_2$

$\neg A_4, A_2, \neg A_3, \neg A_5, A_1$

Search in modal logic:

Two alternating orthogonal components of search:

– Modal search: model spanning
  - jumping among states
  - conjunctive branching
  - up to linearly many successors
– Propositional search: local search
  - reasoning within the single states
  - disjunctive branching
  - up to exponentially many successors



Some Systems

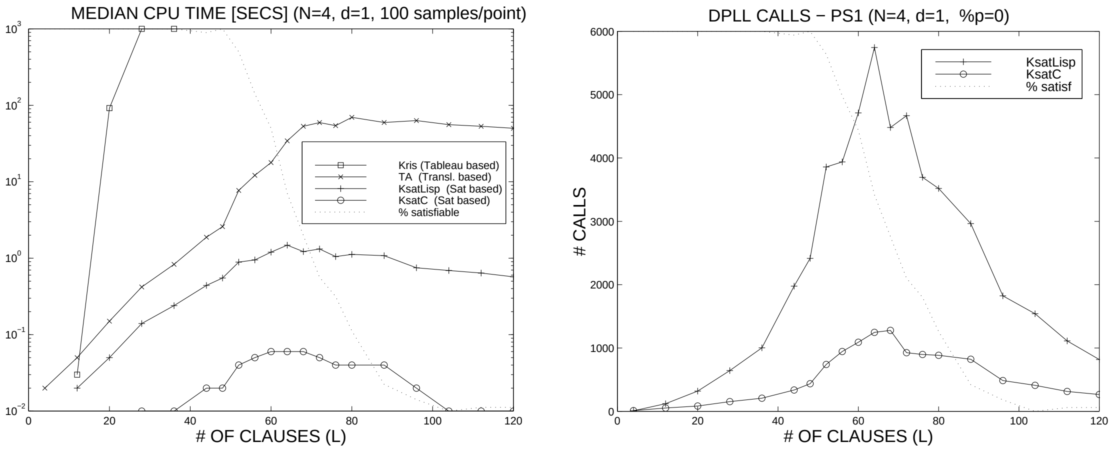
– Kris [10], CRACK [18],
  - Logics: $\mathcal{ALC}$ & many description logics
  - Boolean reasoning technique: semantic tableau
  - Optimizations: preprocessing
– K-SAT [41, 36]
  - Logics: K(m), $\mathcal{ALC}$
  - Boolean reasoning technique: DPLL
  - Optimizations: preprocessing, early pruning

Some Systems (cont.)

- FaCT & DLP [48]
  - Logics: $\mathcal{ALC}$ & many description logics
  - Boolean reasoning technique: DPLL-like
  - Optimizations: preprocessing, memoizing, backjumping + optimizations for description logics
- ESAT &*SAT [37]
  - Logics: non-normal modal logics, K(m), $\mathcal{ALC}$
  - Boolean reasoning technique: DPLL
  - Optimizations: preprocessing, early pruning, memoizing, backjumping, learning

---

Some empirical results [36]

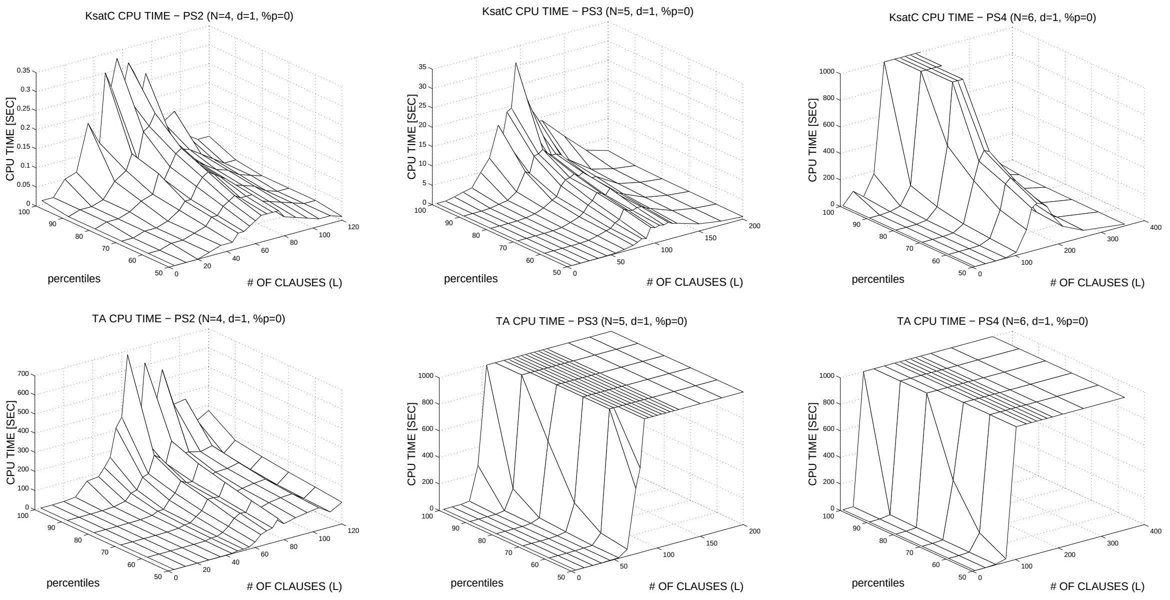


Left: KRIS, TA, K-SAT (LISP), K-SAT (C) median CPU time, 100 samples/point.

Right: K-SAT (LISP), K-SAT (C) median number of consistency checks, 100 samples/point.

Background: satisfiability percentage.

Some empirical results (cont.)



K-SAT (up) versus TA (down) CPU times.

Some empirical results [49]

Formulas of Tableau'98 competition [47]

	branch		*d4*		*dum*		*grz*		*lin*		*path*		*ph*		*poly*		*t4p*	
K	*p*	*n*	*p*	*n*	*p*	*n*	*p*	*n*	*p*	*n*	*p*	*n*	*p*	*n*	*p*	*n*	*p*	*n*
leanK 2.0	1	0	1	1	0	0	0	≥21	≥21	4	2	0	3	1	2	0	0	0
□KE	13	3	13	3	4	4	3	1	≥21	2	17	5	4	3	17	0	0	3
LWB 1.0	6	7	8	6	13	19	7	13	11	8	12	10	4	8	8	11	8	7
TA	9	9	≥21	18	≥21	≥21	≥21	≥21	≥21	≥21	20	20	6	9	16	17	≥21	19
***SAT 1.2**	≥21	12	≥21	≥21	≥21	≥21	≥21	≥21	≥21	≥21	≥21	≥21	8	12	≥21	≥21	≥21	≥21
Crack 1.0	2	1	2	3	3	≥21	1	≥21	5	2	2	6	2	3	≥21	≥21	1	1
Kris	3	3	8	6	15	≥21	13	≥21	6	9	3	11	4	5	11	≥21	7	5
Fact 1.2	6	4	≥21	8	≥21	≥21	≥21	≥21	≥21	≥21	7	6	6	7	≥21	≥21	≥21	≥21
DLP 3.1	19	13	≥21	≥21	≥21	≥21	≥21	≥21	≥21	≥21	≥21	≥21	7	9	≥21	≥21	≥21	≥21

	45		*branch*		*dum*		*grz*		*md*		*path*		*ph*		*poly*		*t4p*	
KT	*p*	*n*	*p*	*n*	*p*	*n*	*p*	*n*	*p*	*n*	*p*	*n*	*p*	*n*	*p*	*n*	*p*	*n*
TA	17	6	13	9	17	9	≥21	≥21	16	20	≥21	16	5	12	≥21	1	11	0
Kris	4	3	3	3	3	14	0	5	3	4	1	13	3	3	2	2	1	7
FaCT 1.2	≥21	≥21	6	4	11	≥21	≥21	≥21	4	5	5	3	6	7	≥21	7	4	2
DLP 3.1	≥21	≥21	19	12	≥21	≥21	≥21	≥21	3	≥21	16	14	7	≥21	≥21	12	≥21	≥21

	45		*branch*		*dum*		*grz*		*md*		*path*		*ph*		*poly*		*t4p*	
S4	*p*	*n*	*p*	*n*	*p*	*n*	*p*	*n*	*p*	*n*	*p*	*n*	*p*	*n*	*p*	*n*	*p*	*n*
KT4	1	6	2	3	0	17	5	8	≥21	18	1	2	2	2	2	2	0	3
leanS4 2.0	0	0	0	0	0	0	1	1	2	2	1	0	1	0	1	1	0	0
□KE	8	0	≥21	≥21	0	≥21	6	4	3	3	9	6	4	3	1	≥21	3	1
LWB 1.0	3	5	11	7	9	≥21	8	7	8	6	8	6	4	8	4	9	9	12
TA	9	0	≥21	4	14	0	6	≥21	9	10	15	≥21	5	5	≥21	1	11	0
FaCT 1.2	≥21	≥21	4	4	2	≥21	5	4	8	4	2	1	5	4	≥21	2	5	3
DLP 3.1	≥21	≥21	18	12	≥21	≥21	10	≥21	3	≥21	15	15	7	≥21	≥21	≥21	≥21	≥21

SAT techniques for modal logics: summary

- SAT techniques have been successfully applied to modal/description logics
- Many optimizations applicable.
- Other approaches:
  - Tableaux approaches [10, 18, 46]
  - F.O. translation methods [50]
  - Inverse methods [92]
  - Automata-theoretic BDD-based methods [66, 67]

Case Study: Mathematical Reasoning

MATH-SAT [3, 95, 21, 60, 7, 6, 9, 84, 29]

- Boolean combinations of mathematical propositions on the reals or integers.
- Typically NP-complete
- Various fields of application: temporal reasoning, scheduling, formal verification, resource planning, etc.

Syntax

Let $\mathcal{D}$ be the domain of either reals $\mathbb{R}$ or integers $\mathbb{Z}$ with its set $\mathcal{OP}_{\mathcal{D}}$ of arithmetical operators.

Given a non-empty set of primitive propositions $\mathcal{A} = \{A_1, A_2, \ldots\}$ and a set $\mathcal{E}_{\mathcal{D}}$ of (linear) mathematical expressions over $\mathcal{D}$, the mathematical language $\mathcal{L}$ is the least set of formulas containing $\mathcal{A}$ and $\mathcal{E}_{\mathcal{D}}$ closed under the set of propositional connectives $\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$.

Syntax: math-terms and math-formulas

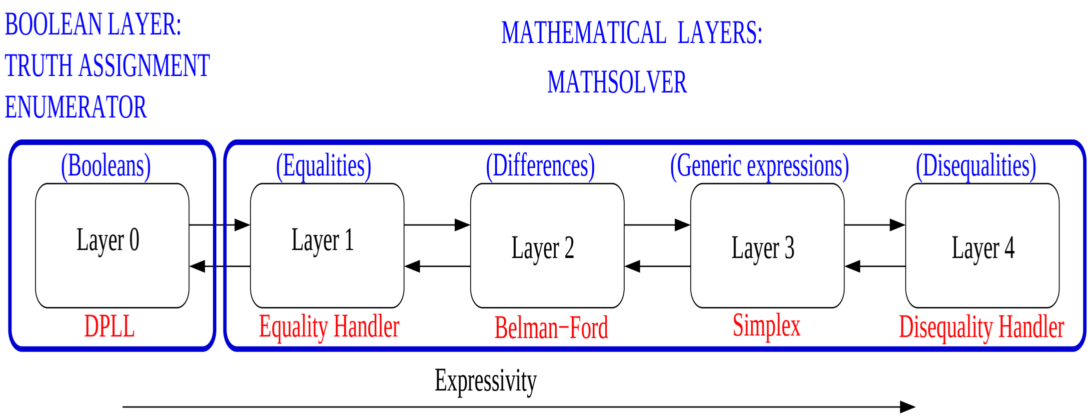
- a constant $c_i \in \mathbb{R}[\mathbb{Z}]$ is a math-term;
- a variable v_i over $\mathbb{R}[\mathbb{Z}]$ is a math-term;
- $c_i \cdot v_j$ is a math-term, $c_i \in \mathbb{R}$ and v_j being a constant and a variable over $\mathbb{R}[\mathbb{Z}]$;
- if t_1 and t_2 are math-terms, then $-t_1$ and $(t_1 \otimes t_2)$ are math-terms, $\otimes \in \{+, -\}$.
- a boolean proposition A_i over $\mathbb{B} := \{\bot, \top\}$ is a math-formula;
- if t_1, t_2 are math-terms, then $(t_1 \bowtie t_2)$ is a math-formula, $\bowtie \in \{=, \neq, >, <, \geq, \leq\}$;
- if φ_1, φ_2 are math-formulas, then $\neg\varphi_1$, $(\varphi_1 \wedge \varphi_2)$, $(\varphi_1 \vee \varphi_2)$, $(\varphi_1 \rightarrow \varphi_2)$ and $(\varphi_1 \leftrightarrow \varphi_2)$, are math-formulas.

Interpretations

Interpretation: a map I assigning real [integer] and boolean values to math-terms and math-formulas respectively and preserving constants and operators:

- $I(A_i) \in \{\top, \bot\}$, for every $A_i \in \mathcal{A}$;
- $I(c_i) = c_i$, for every constant $c_i \in \mathbb{R}$;
- $I(v_i) \in \mathbb{R}$, for every variable v_i over $\mathbb{R}$;
- $I(t_1 \otimes t_2) = I(t_1) \otimes I(t_2)$, for all math-terms t_1, t_2 and $\otimes \in \{+, -, \cdot\}$;
- $I(t_1 \bowtie t_2) = I(t_1) \bowtie I(t_2)$, for all math-terms t_1, t_2 and $\bowtie \in \{=, \neq, >, <, \geq, \leq\}$;
- $I(\neg\varphi_1) = \neg I(\varphi_1)$, for every math-formula φ_1;
- $I(\varphi_1 \wedge \varphi_2) = I(\varphi_1) \wedge I(\varphi_2)$, for all math-formulas φ_1, φ_2.

MATH-SAT: A Layered Architecture [6]



- ▷ Organized in layers of increasing expressive power,
- ▷ Specialized algorithms for particular propositions
- ▷ Each layer comes into play only when needed

Motivating application domains

- Propositional bounded model checking (BMC) [15]: layer 0
  purely propositional atoms $A_1, A_2, \ldots$
- BMC for timed system (no-loops) [9]: layers 0-2
  atoms in the form $A_i, (x = y), (x - y \leq C)$
- BMC for timed (with-loops) and hybrid systems [9, 8]: layers 0-3
  atoms in the form $A_i, (x = y), (x - y \leq C), (x - y = z - w)$
- ...

Layer 0: modified DPLL (SIM)

```
bool MATH-SAT($\varphi, \mu$)
    if ($\varphi == \top$)                                  /* base         */
        then return (MATH-SOLVE($\mu$)==satisfiable);
    if ($\varphi == \bot$)                                  /* backtrack */
        then return False;
    if {a unit clause $(l)$ occurs in $\varphi$}            /* unit        */
        then return MATH-SAT($assign(l, \varphi), \mu \wedge l$);
    if Likely-Unsatisfiable($\mu$)                          /* early pruning */
        if (MATH-SOLVE($\mu$) == False)
            then return False;
    l = choose-literal($\varphi$);                          /* split        */
    return    MATH-SAT($assign(l, \varphi), \mu \wedge l$)   or
              MATH-SAT($assign(\neg l, \varphi), \mu \wedge \neg l$);
```

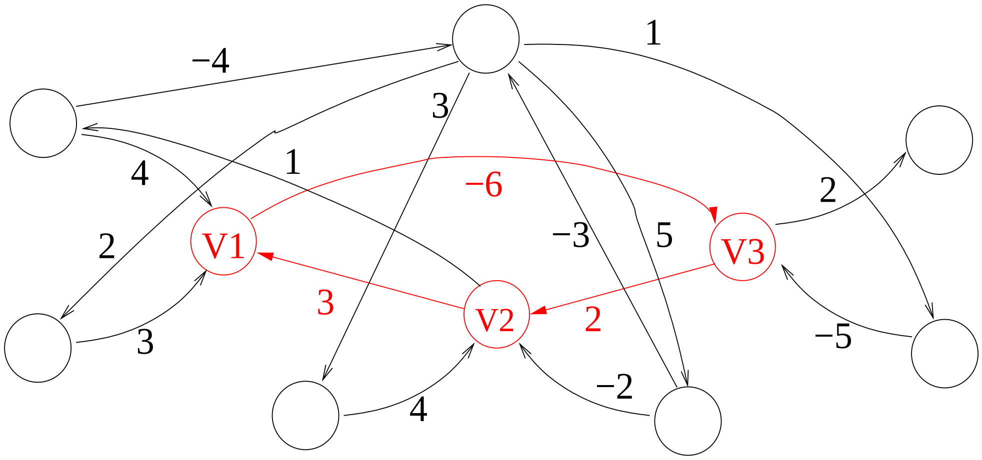
Layer 1: Eliminating Equalities

1. Reveal equalities. Build equivalence classes.
   E.g.: $\{\underline{(v_i = v_j)}, \underline{(v_j = v_k)}, (v_i - v_j \leq 3), (v_i - v_k \leq -2), \ldots\}$
2. Eliminate equivalences and substitute variables:
   $\Longrightarrow \{\ldots, (v_k - v_k \leq 3), (v_k - v_k \leq -2), \ldots\}$
3. Remove all valid atoms, reveal inconsistent atoms:
   - Return "false" if there are inconsistent atoms.
     $\{\ldots, \underline{(v_k - v_k \leq -2)}, \ldots\} \Longrightarrow$ false
   - Invoke layer 2 on the resulting set otherwise.

Layer 2: Handling differences

▷ Deals with "difference" atoms: $(x - y \leq C)$
$\{\ldots, (v_1 - v_2 \leq 3), (v_2 - v_3 \leq 2), (v_3 - v_1 \leq -6), \ldots\}$

▷ Use Belman-Ford's minimal path algorithm with negative cycle detection



Layer 3: dealing with other linear expressions

E.g., $\{(x - y = z - w), (2x - 3y + 4z \leq 5), \ldots\}$

- ▷ Invoke a Simplex Algorithm

Layer 4: Dealing with disequalities

- ▷ Unnecessary with the problems of our interest
  (BMC for timed & hybrid systems)
- ▷ Lazy approach:
  1. Use levels 1, 2, 3 ignoring disequalities. If unsatisfiable, return false.
  2. If the interpretation found verifies the disequalities, return it.
  3. Otherwise, split the two subcases and restart
     $(x \neq y) \Longrightarrow (x < y) \vee (x > y)$
- ▷ Alternative: expand into a disjunction, add mutex clauses

$$\begin{array}{lll} \ldots \vee (x < y) \vee (x > y) & \wedge & instead\ of\ \text{“}\ldots \vee \neg(x = y)\text{''} \\ \neg(x < y) \vee \neg(x > y) & \wedge & \\ \neg(x = y) \vee \neg(x < y) & \wedge & iff\ (x = y)\ occurs\ positively \\ \neg(x = y) \vee \neg(x > y) & & \quad '' \quad '' \quad '' \quad '' \end{array}$$

Some Systems

- Tsat [3]
  - Logics: disjunctions of difference expressions (positive math-atoms only)
  - Applications: temporal reasoning
  - Boolean reasoning technique: DPLL
  - Optimizations: preprocessing, static learning, forward checking
- LPsat [95]
  - Logics: MATH-SAT (positive math-atoms only)
  - Applications: resource planning
  - Boolean reasoning technique: DPLL
  - Optimizations: preprocessing, backjumping, learning, triggering

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Some systems (cont.)

- DDD [60]
  - Logics: boolean + difference expressions
  - Applications: formal verification of timed systems
  - Boolean reasoning technique: OBDD
  - Optimizations: preprocessing, early pruning
- MATH-SAT [6]
  - Logics: MATH-SAT
  - Applications: resource planning, formal verification of timed systems
  - Boolean reasoning technique: DPLL
  - Optimizations: preprocessing, enhanced early pruning, backjumping, learning, triggering

Some systems (cont.)

- CVC [12, 89]
  - Logics: boolean + linear real arithmetic + arrays + inductive datatypes
  - Applications: formal verification
  - Boolean reasoning technique: DPLL
  - Optimizations: backjumping,learning
- ICS [78, 31]
  - Logics: boolean + linear real arithmetic + arrays
  - Applications: formal verification
  - Boolean reasoning technique: DPLL
  - Optimizations: backjumping,learning
- ...

Related systems

Other related systems:

- RDL [4]
- Simplify [64]
- STeP [16]
- UCLID [77]
- ...

SAT + mathematical reasoning: summary

- SAT techniques have been successfully applied to MATH-SAT
- Many optimizations applicable.
- Currently competitive with state-of-the-art applications for temporal reasoning, resource planning, formal verification of timed systems, formal verification of circuits at abstract level.

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- Combination Methods in Automated Reasoning
  `http://combination.cs.uiowa.edu/`
- SMT-LIB - The Satisfiability Modulo Theories Library
  `http://goedel.cs.uiowa.edu/smtlib/`
- SATLive! - Up-to-date links for SAT
  `http://www.satlive.org/index.jsp`
- SATLIB - The Satisfiability Library
  `http://www.intellektik.informatik.tu-darmstadt.de/SATLIB/`