Tight Bounds on Maximal and Maximum Matchings

Therese Biedl[1], Erik D. Demaine[1], Christian A. Duncan[2], Rudolf Fleischer[3], and Stephen G. Kobourov[4]

[1] Department of Computer Science, Waterloo University
Waterloo, Canada
{biedl,eddemaine}@uwaterloo.ca

[2] Department of Computer Science, University of Miami
Miami, FL, USA
duncan@cs.miami.edu

[3] Department of Computer Science,
The Hong Kong University of Science and Technology, Hong Kong
rudolf@cs.ust.hk

[4] Department of Computer Science, University of Arizona
Tucson, AZ, USA
kobourov@cs.arizona.edu

Abstract. In this paper, we study bounds on maximal and maximum matchings in special graph classes, specifically triangulated graphs and graphs with bounded maximum degree. For each class, we give a lower bound on the size of matchings, and prove that it is tight for some graph within the class.

1 Introduction

The problem of finding a maximum matching in a graph has a long and distinguished history beginning with the early work of Petersen [11], König [9], Hall [6], and Tutte [13]. The fastest algorithms to find a maximum matching in an n-vertex m-edge graph takes $O(\sqrt{n}m)$ time, for bipartite graphs [7] as well as for general graphs [10].

One intensely studied topic is whether a graph has a perfect matching, i.e., a matching of size $n/2$. This was shown for 3-regular biconnected graphs [11] and for k-regular bipartite graphs [9], and the perfect matching can be found efficiently for these graphs [2,12,4]. Tutte [13] characterized when a graph has a perfect matching, but no algorithm that can find a perfect matching in an arbitrary graph faster than finding a maximum matching is known.

Not as much is known about bounds for graphs that do not have a perfect matching. Recently, Duncan, Goodrich and Kobourov [5] showed that any planar triangulated graph has a matching of size $\frac{n}{12}$ that satisfies additional constraints.

Our research was originally motivated by the question whether the bound of $\frac{n}{12}$ in [5] could be improved by dropping the extra constraints. Thus, we studied the size of maximal and maximum matchings in planar triangulated graphs. (We

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included maximal matchings because such matchings can be computed easily in linear time.)

It is known that every triangulated planar graph without separating triangles is 4-connected, hence has a Hamiltonian cycle [14], and hence a matching of size $\lfloor \frac{n}{2} \rfloor$. As we will see, we can generalize this to all triangulated planar graphs by including the number of separating triangles (or more precisely, the number of leaves in the tree of 4-connected components) in the bound on the matching.

Next, we study graphs with small maximum degree. It is known that every 3-regular biconnected graph has a perfect matching [11]. As we will see, we can generalize this to all graphs with maximum degree 3 by including the number of cutvertices (or more precisely, the number of leaves in the tree of 2-connected components) and the number of vertices of smaller degree. The proof for maximal matchings generalizes even further to graphs of maximum degree k.

An overview of our results is given in Table 1. All entries are lower bounds on the size of the matching of a certain type. Also, all bounds are tight for some graph within this class. We typically give two bounds: one bound that depends only on n or m, and one bound that also includes other parameters of the graph.

Table 1. Overview of the results in this paper. Here ℓ_4 denotes the number of leaves in the 4-block tree, ℓ_2 denotes the number of leaves in the 2-block tree, and n_2 denotes the number of vertices of degree 2 (see Section 2 for precise definitions). All bounds in the table are tight

Graph	Matching type	Bound 1	Bound 2
Triangulated planar	Maximal	$\frac{n+4}{6}$	$\frac{n+2}{4} - \frac{\ell_4}{6}$
	Maximum	$\frac{n+4}{3}$	$\frac{n}{2} - \frac{\ell_4}{4} + 1$
Max-deg k	Maximal	$\frac{m}{4k-2}$	$\frac{m}{4k-2}$
Max-deg 3	Maximum	$\frac{n-1}{3}$	$\frac{n}{2} - \frac{\ell_2}{3} - \frac{n_2}{6}$
3-regular	Maximum	$\frac{4n-1}{9}$	$\frac{n}{2} - \frac{\ell_2}{3}$

2 Definitions

Let $G = (V, E)$ be a graph with vertices V and edges E, we denote $|V| = n(G) = n$ and $|E| = m(G) = m$. Denote by n_i the number of vertices of degree i, i.e., with exactly i incident edges. We call G *3-regular* if every vertex has degree 3, and a *max-deg-k graph* if every vertex has degree at most k. G is called *simple* if there are no loops and no multiple edges, and *connected* if for any pair of vertices there exists a path from one vertex to the other. In this paper, we assume that G is simple and connected.

A connected graph G is called *k-connected* if for any set C of at most $k-1$ vertices, the graph that results from deleting the vertices in C is still connected.

A 2-connected graph is also called *biconnected*. If a connected graph is not biconnected, then it must have a vertex v such that $G - v$ is not connected; such a vertex is called a *cutvertex*. If G has cutvertices, then its *biconnected components* are the maximal biconnected subgraphs of the graph. The *2-block tree* is obtained by defining one node for every biconnected component and one node for every cutvertex, and connecting two nodes if and only if one is a cutvertex contained in the biconnected component of the other node. As the name suggests, the 2-block tree is a tree. Let $\ell_2(G)$ denote the number of *leaves* in the 2-block tree; we write ℓ_2 if the graph in question is clear.

A *planar graph* is a graph that can be drawn in the plane without a crossing. Such a planar drawing divides the plane into connected pieces called *faces*. The *degree* of a face is the number of times that a vertex is incident to a boundary of a face. In a simple planar graph with at least three vertices, every face has degree at least 3. A planar graph is called *triangulated* if all faces have degree 3 (i.e., they are a 3-cycle, also called a *triangle*). A triangulated graph has exactly $3n - 6$ edges and is 3-connected.

A *separating triangle* in a planar graph is a triangle that is not the boundary of a face, i.e., a triangle such that there are vertices both inside and outside the triangle. Assume that G is a triangulated graph that is not 4-connected. Then there exist three vertices $\{u, v, w\}$ such that removing them splits G into at least two parts. Since G is triangulated, $\{u, v, w\}$ must form a separating triangle. Hence a triangulated graph is 4-connected if and only if it has no separating triangle.

If G is a triangulated graph that is not 4-connected, then we can split it into its 4-connected components as follows. We say that a separating triangle T_1 is *inside* another separating triangle T_2 if none of the vertices of T_2 is in the outside of T_1. (Note that some vertices may be in both triangles.) Let $T_1, \ldots, T_k$ be those separating triangles that are not inside any other separating triangle. Denote by G^0 the graph that results from G by deleting all vertices that are inside $T_1, \ldots, T_k$; then G^0 is a 4-connected graph. For $i = 1, \ldots, k$, denote by G_i the graph that results from taking all vertices inside T_i and adding the vertices of T_i. Recursively compute the 4-connected components of $G_1, \ldots, G_k$; these are also 4-connected components of G.

Following the construction of 4-connected components, one can obtain a tree to store these components. The root of this tree is G^0, the 4-connected graph that results from deleting the insides of the separating triangles. This component has one child for each separating triangle that is not inside another separating triangle. The subtrees of these children are computed recursively from $G_1, \ldots, G_k$. This so-called *4-block tree* tree can be computed in $O(n)$ time [8]. Denote by $\ell_4(G)$ the number of leaves of the 4-block tree; we write ℓ_4 if the graph in question is clear.

A *matching* is a set M of edges such that no vertex has two or more incident edges in M. For a given matching M, define V_M to be the *matched vertices*, i.e., the vertices with an incident edge in M, and V_U to be the *unmatched vertices*, i.e., $V - V_M$. A matching is called a *maximal matching* if there is no edge between two unmatched vertices, i.e., we cannot add an edge to the matching. A matching is called a *maximum matching* if it has the maximum possible cardinality among all matchings. A *perfect matching* is a matching that leaves no unmatched vertices, i.e., a matching with $n/2$ edges.

2.1 Tutte's Theorem and Berge's Generalization

In 1947, Tutte [13] proved a characterization of the existence of a perfect matching. His theorem uses the concept of *odd components* which is explained by the following. Let T be an arbitrary subset of vertices. Removing T from the graph may split the graph into a number of connected components. Some of those may have an even number of vertices, and some may have an odd number of vertices. We denote by $o(T)$ the number of components of $G-T$ that have an odd number of vertices; these are also called the *odd components*.

Tutte proved that a graph has a perfect matching if and only if for any vertex set T, the number of odd components of T is not bigger than T. Berge showed in 1957 how to extend this theorem to characterize the size of a maximum matching, again using vertex sets T and their odd components.

Lemma 1. *[13,1] Let G be a graph. For any set $T \subset V$, any matching contains at least $o(T) - |T|$ unmatched vertices. Moreover, there exists a set $T \subset V$ such that any maximum matching of G contains exactly $o(T) - |T|$ unmatched vertices.*

3 Triangulated Graphs

Duncan, Goodrich and Kobourov [5] proved that any planar triangulated graph has a matching with at least $n/12$ edges that do not belong to any separating triangle. It trivially follows that any triangulated planar graph has a matching of size at least $n/12$. We will here obtain better bounds by dropping the condition on separating triangles.

3.1 Maximal Matching for Triangulated Graphs

First we study maximal matchings. We initially give a bound that depends on the number of leaves in the 4-block tree, and then estimate the number of such leaves. We need an easy observation about the relationship between face sizes and vertices in a planar graph.

Lemma 2. *Assume that G is a planar graph with $n \geq 3$ vertices, that has f_3 faces of degree 3 and f_4 faces of degree at least 4. Then $f_3 + 2f_4 \leq 2n - 4$.*

Proof: We have $3f_3 + 4f_4 \leq 2m$, since the left-hand side counts each edge at most twice. Also, $m \leq 3n - 6 - f_4$, because a triangulated graph has $3n - 6$ edges, and there is at least one missing edge for every face of degree at least 4. Combining the two inequalities gives $3f_3 + 4f_4 \leq 6n - 12 - 2f_4$, which after rearranging yields the result. □

Lemma 3. *Any maximal matching of a planar triangulated graph with at least 4 vertices has size at least* $\frac{n+2}{4} - \frac{1}{8}\ell_4$*, where* ℓ_4 *is the number of leaves in the 4-block tree of the graph.*

Proof: Let M be an arbitrary maximal matching, and let V_M and V_U be the matched and unmatched vertices. Let G_M be the graph induced by the matched vertices. Since G has at least 4 vertices, it must have at least four matched vertices, so $|V_M| \geq 4$ and G_M has no faces of degree less than 3.

In any face of G_M, there can be at most one unmatched vertex of G, for if there were two or more unmatched vertices, then (because G is triangulated) there must be an edge between them, contradicting the maximality. We split the unmatched vertices into two groups: V_U^3 denotes those that are inside a face of G_M of degree 3, whereas V_U^4 denotes those that are inside a face of G_M of degree at least 4. Note that $|V_U^3| \leq \ell_4$, because if there is a vertex inside a triangular face of G_M, then this triangle contains a vertex inside and also a vertex outside (by $n(G_M) \geq 4$), hence it is a separating triangle in G and contains a leaf of the 4-block tree.

By Lemma 2, we have $|V_U^3| + 2|V_U^4| \leq 2n(G_M) - 4 = 2|V_M| - 4$. Since $|V_U^3| + |V_U^4| + |V_M| = n$, we can reformulate this further as $|V_U^3| + 2|V_U^4| = 2(n - |V_U^3| - |V_U^4|) - 4$. Therefore $3|V_U^3| + 4|V_U^4| \leq 2n - 4$, and

$$|V_U| \leq \frac{1}{4}(3|V_U^3| + 4|V_U^4|) + \frac{1}{4}|V_U^3| \leq \frac{1}{4}(2n-4) + \frac{1}{4}\ell_4,$$

which implies $|V_M| \geq n - \frac{n-2}{2} - \frac{1}{4}\ell_4 = \frac{n+2}{2} - \frac{1}{4}\ell_4$ as desired. □

Now we need a bound on ℓ_4. Kant [8] stated that every planar triangulated graph has at most $n-4$ separating triangles. Since he did not prove this claim, we give a proof here for completeness' sake.

Lemma 4. *Any planar triangulated graph has at most* $n-4$ *separating triangles.*

Proof: The proof is by induction. If a graph has a separating triangle, then it must have the three vertices of the triangle and one vertex both inside and outside, so $n \geq 5$, and a graph with 5 vertices can have only one separating triangle. Assume the claim holds for all values up to $n-1$, $n \geq 6$. Let G be any graph of n vertices, and assume it has a separating triangle $\{u, v, w\}$; otherwise we are done. Let G_i and G_o be the graphs inside and outside $\{u, v, w\}$, respectively. We have $n(G_i) + n(G_o) = n + 3$. Both graphs have fewer vertices than G, and so by induction have at most $n(G_i) - 4$ and $n(G_o) - 4$ separating triangles, respectively. Hence, the number of separating triangles in G is at most $1 + (n(G_i) - 4) + (n(G_o) - 4) = n - 4$. □

This bound is tight, see for example the graph class $\mathcal{H}$ that was defined in [DGK99]. Combining this bound with Lemma 6, we obtain that any planar triangulated graph has a matching of size at least $\frac{n+3}{4}$. However, we can do better by obtaining a bound on the number of leaves in the 4-block tree.

Lemma 5. *Any planar triangulated graph has at most* $\frac{2}{3}(n-2)$ *leaves in the 4-block tree.*

Proof: As before, denote by ℓ_4 the number of leaves in the 4-block tree. Let $T_1, \ldots, T_{\ell_4}$ be those separating triangles that form the leaves, and let G_L be the graph induced by the vertices in $T_1, \ldots, T_{\ell_4}$. For each triangle T_i, there is at least one vertex inside T_i that does not belong to any of the other triangles, and therefore not to G_L. Hence $n(G_L) \leq n - \ell_4$.

Every triangle T_i is a face of G_L (because these triangles are leaves of the 4-block tree). But the number of faces in G_L is at most $2n(G_L) - 4$ since G_L is planar. Hence $\ell_4 \leq 2n(G_L) - 4 \leq 2(n - \ell_4) - 4$, or $3\ell_4 \leq 2n - 4$, which yields the result. $\square$

Combining this bound with the bound of Lemma 3, we obtain that every maximal matching of a triangulated planar graph has size at least $\frac{n+2}{4} - \frac{1}{8} \cdot \frac{2}{3}(n-2) = \frac{n+4}{6}$.

Theorem 1. *Any maximal matching of a triangulated planar graph with at least 4 vertices has size at least* $\frac{n+4}{6}$.

The above bound is tight, i.e., there exists a planar triangulated graph with a maximal matching of size $(n+4)/6$. To see this, take any planar triangulated graph G that has a perfect matching M, and add into each face of G one more vertex connected to the three neighbors. Call the resulting graph G', and its number of vertices n'. Then $n' = n + 2n - 4$ (because G has $2n - 4$ faces). Also, M is a maximal matching in G' and $|M| = n/2 = (n'+4)/6$.

3.2 Maximum Matching for Triangulated Graphs

In this section, we provide a bound on the size of a maximum matching in a planar triangulated graph. As we will see, this bound will again depend on the number of leaves in the 4-block tree.

Lemma 6. *Any planar triangulated graph G has a matching of size at least* $\min\{\lfloor \frac{n}{2} \rfloor, \frac{n}{2} - \frac{\ell_4}{4} + 1\}$, *where ℓ_4 is the number leaves of the 4-block tree of G.*

Proof: Let M be a maximum matching, and let T be a vertex set such that there are $o(T) - |T|$ unmatched vertices, i.e., $|V_U| = o(T) - |T|$ (Lemma 1). The claim holds if $|T| \leq 2$, because then $o(T) \leq 1$ since G is 3-connected, and there is at most one unmatched vertex. The claim also holds if $|T| = 3$, because then there are at most two odd components (the inside and the outside of the separating triangle). So we may assume that $|T| \geq 4$.

Let G_T be the graph that is induced by the vertices of T. Observe that no two odd components can be within the same face of G_T since G is triangulated. Let $o_3(T)$ and $o_4(T)$ be the number of odd components that are inside a face of G_T of degree 3 and degree at least 4, respectively. Note that $o_3(T) \leq \ell_4$, because if there is an odd component inside a triangular face of G_T, then this triangle contains a vertex inside and also a vertex outside (by $|T| \geq 4$), hence it is a separating triangle in G and contains a leaf of the 4-block tree.

By Lemma 2, we know that $o_3(T) + 2o_4(T) \leq 2n(G_T) - 4 = 2|T| - 4$, and

$$\begin{aligned} |V_U| &= o(T) - |T| = o_3(T) + o_4(T) - |T| \\ &= \frac{1}{2}(o_3(T) + 2o_4(T)) + \frac{1}{2}o_3(T) - |T| \\ &\leq \frac{1}{2}(2|T| - 4) + \frac{\ell_4}{2} - |T| = \frac{\ell_4}{2} - 2. \end{aligned}$$

So $|V_M| \geq n - |V_U| \geq n - \frac{\ell_4}{2} + 2$ as desired. □

Combining Lemma 6 with Lemma 5, we obtain the bound on maximum matching in triangulated planar graphs.

Theorem 2. *Every planar triangulated graph with at least* 10 *vertices has a matching of size at least* $\frac{n+4}{3}$.

Proof: There is nothing to prove for $n \geq 10$ if G has a matching of size $\lfloor \frac{n}{2} \rfloor$. Otherwise, G has a matching of size $\frac{n}{2} - \frac{\ell_4}{4} + 1 \geq \frac{n}{2} - \frac{1}{6}(n-2) + 1 = \frac{n+4}{3}$. □

The above bound is tight, i.e., there exists a graph for which any matching has at most $(n+4)/3$ edges. This graph is defined for for any $n \equiv 2 \bmod 3$, $n \geq 11$ and shown in Figure 1. It consists of a cycle with $(n-2)/3$ vertices, two vertices connected to each vertex of the cycle (these parts are shown in black), and one more vertex in every face of the above graph (this part is shown in white).

Let T be the $(n+4)/3$ black vertices. Since there are no edges between white vertices, graph $G - T$ has $(2n-4)/3$ isolated vertices, which each form an odd component, so $o(T) - |T| = (2n-4)/3 - (n+4)/3 = (n-8)/3$. Hence in any matching M of the graph, at least $(n-8)/3$ vertices are unmatched and at most $(2n+8)/3$ vertices are matched, so $|M| \leq (n+4)/3$.

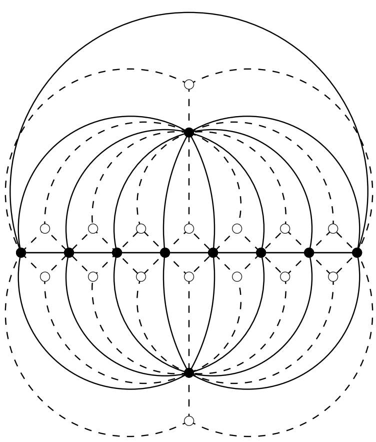

Fig. 1. A planar triangulated graph with a maximum matching of size $\frac{n+4}{3}$

Note also that this graph has $\ell_4 = \frac{2}{3}(n-2)$ separating triangles which are all leaves of the 4-block tree, so Lemma 5 is tight as well.

4 Graphs with Maximum Degree k

Now we devote our attention to another graph class with a special structure; graph with maximum degree k. We chose this graph class because 3-regular biconnected graphs are known to have a perfect matching, and we tried to generalize this to graphs with bounded maximum degree.

Theorem 3. *Any maximal matching of a max-deg-k graph has size at least* $m/(4k-2)$.

Proof: Let M be an arbitrary maximal matching, and let V_M and V_U be the matched and unmatched vertices. We split V_U into k sets, $V_U^i, i = 1, \ldots, k$, where V_U^i is the set of unmatched vertices with degree i.

Let E_U be the set of edges with at least one endpoint in V_U. Recall that since M is maximal no edge can have both endpoints in V_U. Therefore, E_U is the set of all edges between vertices in V_U and vertices in V_M, and $|E_U| = \sum_{i=1}^{k} i|V_U^i|$. Since every vertex in V_M is incident to at most k vertices, and at least one of them is also in V_M, we get $|E_U| \leq (k-1)|V_M|$. Combining, we have

$$\begin{aligned}
|V_U| &= \sum_{i=1}^{k} |V_U^i| \\
&= \sum_{i=1}^{k} \frac{i}{k}|V_U^i| + \sum_{i=1}^{k} \frac{k-i}{k}|V_U^i| \\
&= |E_U|/k + \sum_{i=1}^{k} \frac{k-i}{k}|V_U^i| \\
&\leq (k-1)|V_M|/k + \sum_{i=1}^{k} \frac{k-i}{k}|V_U^i| \\
&\leq (k-1)|V_M|/k + \sum_{i=1}^{k} \frac{k-i}{k}n_i
\end{aligned}$$

Solving for V_M we get

$$|V_M| = n - |V_U| \geq n - (k-1)|V_M|/k - \sum_{i=1}^{k} \frac{k-i}{k}n_i$$

and therefore

$$|V_M| \geq \frac{k}{2k-1}(n - \sum_{i=1}^{k} \frac{k-i}{k}n_i) = \frac{k}{2k-1}(n - \sum_{i-1}^{k} n_i + \frac{1}{k}\sum_{i=1}^{k} i\, n_i) = \frac{m}{2k-1},$$

which yields the result. □

This bound is tight, as illustrated in the graph in Figure 2. The bold edges indicate a maximal matching of size $\frac{m}{4k-2}$.

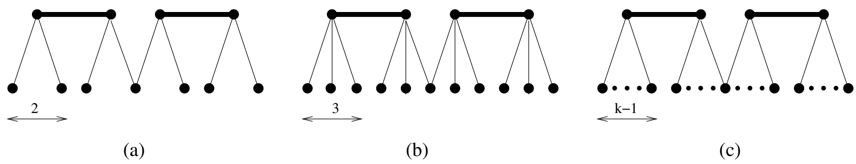

Fig. 2. A max-deg-k graph with a maximal matching of size $m/(4k-2)$ for (a) $k = 3$, (b) $k = 4$, (c) arbitrary k

4.1 Maximum Matching for Max-deg-3 Graphs

We have not succeeded in obtaining a better bound for a *maximum* matching in a graph with maximum degree k, except when $k = 3$.

Lemma 7. *Any max-deg-3 graph G has a matching of size at least $\frac{n}{2} - \frac{\ell_2}{3} - \frac{n_2}{6}$, where ℓ_2 is the number of leaves in the 2-block tree of G and n_2 is the number of vertices of degree 2.*

Proof: Let M be a maximum matching, and let T be a vertex set such that there are $o(T) - |T|$ unmatched vertices (Lemma 1). We define the following three quantities: $o_1(T), o_2(T)$, and $o_3(T)$ are the number of odd components joined to T by one edge, two edges, and at least three edges, respectively. Every odd component joined to T by one edge contains a leaf of the 2-block tree, so $o_1(T) \leq \ell_2$. Every odd component joined to T by two edges must contain at least one vertex of degree 2 (otherwise there would be an odd number of vertices of odd degree), so $o_2(T) \leq n_2$.

The number of edges incident to T is at least $o_1(T) + 2o_2(T) + 3o_3(T)$, but also at most $3|T|$ since G has maximum degree 3. Therefore

$$|V_U| = o(T) - |T| \leq \frac{1}{3}(o_1(T) + 2o_2(T) + 3o_3(T)) + \frac{2}{3}o_1(T) + \frac{1}{3}o_2(T) - |T|$$
$$\leq \frac{1}{3}(3|T|) + \frac{2}{3}\ell_2 + \frac{1}{3}n_2 - |T| = \frac{2}{3}\ell_2 + \frac{1}{3}n_2,$$

and $|V_M| \geq n - \frac{2}{3}\ell_2 - \frac{n_2}{3}$, which proves the claim. □

To obtain a bound that only depends on n, we need to bound ℓ_2 and n_2.

Lemma 8. *Every max-deg-3 graph has $2\ell_2 + n_2 \leq n + 2$, where ℓ is the number of leaves in the 2-block tree and n_2 is the number of vertices of degree 2.*

Proof: Let G be a connected max-deg-3 graph. If all leaves of the 2-block tree of G contain a vertex of degree 1, then $\ell_2 = n_1$ (because every vertex of degree 1 implies a leaf). A simple counting argument shows that $n_1 \leq n_3 + 2$, and hence $2\ell_2 + n_2 = 2n_1 + n_2 \leq n_1 + n_2 + n_3 + 2 = n + 2$.

If some leaves of the 2-block tree of G do not contain a vertex of degree 1, then obtain a new graph G' by deleting from these leaves all vertices except the cutvertex. Note that G and G' have equally many leaves of the 2-block tree, and G' has at most as many vertices of degree 2 as G. Since the claim holds for G', we have $2\ell_2(G) + n_2(G) \leq 2\ell_2(G') + n_2(G') \leq n(G') + 2 \leq n(G) + 2$. □

Combining this lemma with Lemma 7, we obtain that the number of unmatched vertices is at most $\frac{2}{3}\ell_2 + \frac{n_2}{3} \leq \frac{n+2}{3}$, hence the maximum matching has size at least $\frac{n-1}{3}$.

Theorem 4. *Every max-deg 3 graph has a matching of size at least* $\frac{n-1}{3}$.

This bound is tight, as can be seen from the graph in Figure 3, for which the maximum matching has size $\frac{n-1}{3}$.

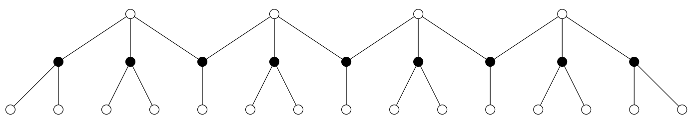

Fig. 3. A max-deg-3 graph for which a maximum matching has size $\frac{n-1}{3}$

One can observe that this graph does not have any vertices of degree 2. However, the factor "$-\frac{n_2}{6}$" in Lemma 7 is tight, as demonstrated by the following example: Consider any 3-regular graph with n vertices and $m = \frac{3}{2}n$ edges. Now split every edge into two, and add a degree-two vertex in the middle. This gives a new graph G' with $n' = n+m = \frac{5}{2}n$ vertices and $n_2' = \frac{3}{2}n$ vertices of degree 2.

A maximum matching of G' has at most $n' - \frac{n}{2}$ matched vertices, because setting T to be the n original vertices, we obtain $\frac{3}{2}n$ odd components from the added vertices. So the maximum matching has size $\frac{n'}{2} - \frac{n}{4} = n$. Since $\frac{n_2'}{6} = \frac{n}{4}$, this proves that the bound of Lemma 7 is tight.

Note, however, that the size of the maximum matching for this graph is $\frac{2}{5}n' > \frac{1}{3}n'$. It remains open whether there exists a better bound on the size of the maximum matching if a graph is forced to have vertices of degree 2, for example whether a bound of $\frac{1}{3}n + \frac{1}{9}n_2$ holds for the size of a maximum matching in a graph with maximum degree 3.

4.2 Maximum Matchings for 3-regular Graphs

For 3-regular graphs we can improve the bounds of Theorem 4 even further.

Lemma 9. *Every 3-regular graph has at most* $\frac{n+2}{6}$ *leaves in the 2-block tree.*

Proof: Let C be a biconnected component that is a leaf in the 2-block tree, and let v be its unique cutvertex. We claim that C has at least 5 vertices, and prove this as follows: Since G is simple, v must have a neighbor $w \neq v$ in C.

Since G is 3-regular, w must have 3 neighbors, which are all in C since w is not a cutvertex. So C has at least 4 vertices. Since all vertices except v in C have odd degree, but v has even degree, C has an odd number of vertices, so C has at least 5 vertices.

Let G_L be the graph that results from G by deleting all vertices that are part of a leaf of the 2-block tree and not a cutvertex. Hence for every leaf we delete at least 4 vertices, so $n(G_L) \leq n - 4\ell_2$. The remaining graph is connected, hence $m(G_L) \geq n(G_L) - 1$. Also, every cutvertex that belonged to a leaf of G has degree 1 in G_L, whereas all other vertices have degree 3, so $2m(G_L) = \ell_2 + 3(n(G_L) - \ell_2)$. Thus we obtain $\ell_2 + 3(n(G_L) - \ell_2) = 2m(G_L) \geq 2n(G_L) - 2$, which implies $n(G_L) \geq 2\ell_2 - 2$, therefore $2\ell_2 \leq n(G_L) + 2 \leq n - 4\ell_2 + 2$ and $\ell_2 \leq \frac{1}{6}(n+2)$. □

Consequently, the maximum matching of a 3-regular graph has size at least $\frac{n}{2} - \frac{\ell_2}{3} = \frac{n}{2} - \frac{1}{18}(n+2) = \frac{1}{9}(4n-1)$.

Theorem 5. *Every 3-regular graph has a matching of size at least* $\frac{4n-1}{9}$.

This bound is also tight, which can be seen by attaching the smallest possible 3-regular graph to every leaf of the graph of Figure 3. The resulting graph (shown in Figure 4) is defined for $n = 16 \bmod 18$. The set of black vertices has size $\frac{n+2}{9}$, and yields $\frac{2n+11}{9}$ odd components. Hence any matching has size at most $\frac{4n-1}{9}$.

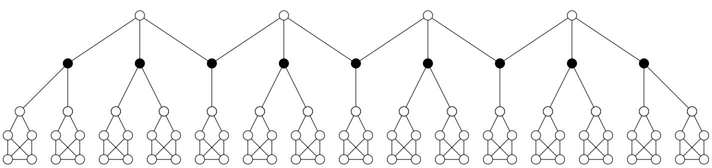

Fig. 4. A simple 3-regular graph with a maximum matching of size $\frac{4n-1}{9}$

5 Conclusion

In this paper, we studied bounds on the size of maximal and maximum matchings in special graphs classes, in particular triangulated planar graphs, graphs with maximum degree k, graphs with maximum degree 3 and 3-regular graphs. We obtain lower bounds on the size of such matchings, and showed that the bounds are tight for some graph within the class.

We leave a number of open problems:

- How quickly can we find matchings that are known to exist? A maximal matching can be found in linear time, but can we find, say, a matching of size $\frac{n}{2} - \frac{\ell_4}{4} + 1$ in a planar triangulated graph in less than $O(m\sqrt{n})$ time?

– What can be said about the size of a maximum matching in a graph with maximum degree k? Can we obtain a bound better than $m/(4k-2)$?
– Is there a graph with maximum degree 3 for which a maximum matching has size $\frac{n}{2} - \frac{\ell_2}{3} - \frac{n_2}{6}$, and which has a significant number of vertices of degree 2? Or if not, can we show a better bound?

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