Gauss and AGM

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Introduction

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derivation of equation.
what has it to do w/ the lemniscate
agm
properties of I
elliptic integrals
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The Elliptic Integral of the First Kind

Define the a *lemniscate*, $r^2 = \cos 2\theta$. In analogy to the unit circle, whose circumference is 2π, the circumference (total arc length) of the lemniscate is $2\tilde{\omega}$, using Gauss' notation. Using the polar form for arc length,

$$\int \sqrt{r^2 + (dr/d\theta)^2}\ d\theta$$

and implicitely differentiating the leminscate equation,

$$r\ dr = -\sin 2\theta\ d\theta$$

the arc length for one quadrant of the lemniscate is,

$$\tilde{\omega}/2 = \int_0^{\pi/4} \sqrt{\cos 2\theta - \sin 2\theta/r}\ d\theta = \int_0^{\pi/4} \frac{d\theta}{\sqrt{\cos 2\theta}}$$

We substitute $\cos 2\theta = \cos^2\phi$. Taking differentials,

$$\sin 2\theta\ d\theta = \cos\phi \sin\phi\ d\phi$$

and then manipulating $\sin 2\theta$,

$$\sin^2 2\theta = 1 - \cos^2 2\theta = 1 - \cos^4\phi = \sin^2\phi(1 + \cos^2\phi)$$

to obtain,

$$d\theta = \cos\phi/\sqrt{1 + \cos^2\phi}\ d\phi$$

The result of the substitution is then,

$$\tilde{\omega}/2 = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1+\cos^2\phi}} = \int_0^{\pi/2} \frac{d\phi}{\sqrt{2\cos^2\phi + \sin^2\phi}}$$

This last integral is an *elliptic integral of the first kind.*

Definition 1 *The Elliptic Integral of the First Kind is defined as,*

$$I(a,b) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}}$$

The Arithmetic-Geometric Mean

Given two reals a and b, their arithmetic-geometric mean AGM(a,b) is the common limit of a and b under the iteration,

$$\begin{array}{ccc} (a+b)/2 & \to & a \\ \sqrt{ab} & \to & b \end{array}$$

The convergence is extremely fast.

Theorem 1 *Let a_o, b_o be the next iterate of a, b in the AGM procedure, $a_o = (a+b)/2$ and $b_o = \sqrt{ab}$. Then $I(a,b) = I(a_o, b_o)$.*

Proof: Gauss's proof is to indicate the remarkable substitution,

$$\sin\phi = \frac{2a\sin\phi'}{a+b+(a-b)\sin^2\phi'}$$

We give this proof in an appendix. Here is a simpler proof by Nick Lord, who seems to credit Schoenberg. We express the integral after the substitution $t = b\tan\phi$,

$$I(a,b) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dt}{\sqrt{(t^2+a^2)(t^2+b^2)}}$$

Substitute $t = (x - ab/x)/2$. Considering in turn each factor in the denominator and the differential,

$$\begin{aligned} t^2 + a_o^2 &= ((x-ab/x)/2)^2 + ((a+b)/2)^2 \\ &= (1/4x^2)(x^4 - 2abx^2 + (ab)^2) + (a^2+b^2+2ab)x^2 \\ &= (1/4x^2)(x^4 + (a^2+b^2)x^2 + (ab)^2) \\ &= (1/4x^2)(x^2+a^2)(x^2+b^2). \end{aligned}$$

$$\begin{aligned} t^2 + b_o^2 &= ((x-ab/x)/2)^2 + (\sqrt{ab})^2 \\ &= (1/4x^2)(x^4 - 2abx^2 + (ab)^2) + 4abx^2 \end{aligned}$$

$$
\begin{aligned}
&= (1/4x^2)(x^4 + 2abx^2 + (ab)^2) \\
&= (1/4x^2)(x^2 + ab)^2.
\end{aligned}
$$

$$
\begin{aligned}
dt &= d(x - ab/x)/2 \\
&= (x^2 + ab)/(2x^2)\, dx
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
I(a_o, b_o) &= \frac{1}{2}\int_{-\infty}^{\infty} \frac{dt}{\sqrt{(t^2 + a_o^2)(t^2 + b_o^2)}} \\
&= \frac{1}{2}\int_{0}^{\infty} \frac{2x}{\sqrt{(x^2 + a^2)(x^2 + b^2)}}\, \frac{2x}{x^2 + ab}\, \frac{x^2 + ab}{2x^2}\, dx \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{\sqrt{(x^2 + a^2)(x^2 + b^2)}} \\
&= I(a, b)
\end{aligned}
$$

The Elliptic Integral of the Second Kind

Definition 2 *Let $J(a,b)$ be the Elliptic Integral of the Second Kind,*

$$
J(a,b) = \int_0^{\pi/2} \sqrt{a^2\cos^2\phi + b^2\sin^2\phi}\; d\phi.
$$

Theorem 2 *Define the special function,*

$$
L(a,b) = \int_0^{\pi/2} \frac{\sin^2\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}}\, d\phi
$$

Then $(a^2 - b^2)L(a,b) = a^2 I(a,b) - J(a,b)$.

Proof:

$$
\begin{aligned}
(a^2 - b^2)L(a,b) &= \int_0^{\pi/2} \frac{(a^2 - b^2)\sin^2\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}} d\phi \\
&= \int_0^{\pi/2} \frac{a^2 - (a^2\cos^2\phi + b^2\sin^2\phi)}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}}\, d\phi \\
&= a^2 I(a,b) - J(a,b)
\end{aligned}
$$

AGM and π

Let $k = \mathrm{AGM}(\sqrt{2}, 1)$, then

$$
\tilde{\omega}/2 = \int_0^{\pi/2} \frac{d\phi}{\sqrt{2\cos^2\phi + \sin^2\phi}} = \int_0^{\pi/2} \frac{d\phi}{\sqrt{k^2\cos^2\phi + k^2\sin^2\phi}} = \pi/(2k).
$$

This results in the remarkable identity, due to Gauss,

$$\pi/\tilde{\omega} = \mathrm{AGM}(\sqrt{2}, 1)$$

Gauss proof of invariance

Gauss was the first to notice and prove that the Elliptic Integral is invariant by substitution of parameters by their AGM. He states simply that the proof is by the substition,

$$\sin\phi = \frac{2a\sin\phi'}{a+b+(a-b)\sin^2\phi'}$$

C. Jacobi gave further guidance by indicating three identities, which we state and prove in the following lemmas. I found that even with this guidance, the proof involves a great deal of algebra. Certainly Gauss saw something in these formulas which lead him rationally along this path.

Lemma 1

$$\cos\phi \quad = \quad \frac{2\cos\phi'\sqrt{a'^2\cos^2\phi' + b'^2\sin^2\phi'}}{a+b+(a-b)\sin^2\phi'}$$

Proof:

$$\cos\phi = \sqrt{1-\sin^2\phi} = \Delta^{-1}\sqrt{\Delta^2 - 4a^2\sin^2\phi'}$$

where,

$$\Delta = a+b+(a-b)\sin^2\phi'$$

so

$$\begin{aligned}
\Delta^2 - 4a^2\sin^2\phi' &= (a+b)^2(\cos^2\phi' + \sin^2\phi') + 2(a+b)(a-b)\sin^2\phi' \\
&\quad +(a-b)^2\sin^4\phi' - 4a^2\sin^2\phi' \\
&= (a+b)^2\cos^2\phi' - (a-b)^2\sin^2\phi' + (a-b)^2\sin^4\phi' \\
&= \cos^2\phi'((a+b)^2 - (a-b)^2\sin^2\phi')
\end{aligned}$$

but

$$\begin{aligned}
(a+b)^2 - (a-b)^2\sin^2\phi' &= (a+b)^2(\cos^2\phi' + \sin^2\phi') - (a-b)^2\sin^2\phi' \\
&= (a+b)^2\cos^2\phi' + 4ab\sin^2\phi'
\end{aligned}$$

Lemma 2

$$\sqrt{a^2\cos^2\phi + b^2\sin^2\phi} = a\frac{a+b-(a-b)\sin^2\phi'}{a+b+(a-b)\sin^2\phi'}$$

Proof: Sustitute $\cos\phi$ and $\sin\phi$ from our identities,

$$a^2\cos^2\phi + b^2\sin^2\phi = \Delta^{-2}a^2\,\Gamma$$

where,

$$\Gamma = 4\cos^2\phi'(a'^2\cos^2\phi' + b'^2\sin^2\phi') + 4b^2\sin^2\phi'$$

To the sum inside the parenthesis, un-prime the constants and preform a trignometric substitution,

$$\begin{aligned} 4(a'^2\cos^2\phi' + b'^2\sin^2\phi') &= (a+b)^2\cos^2\phi' + 4ab\sin^2\phi' \\ &= (a+b)^2(1-\sin^2\phi') + 4ab\sin^2\phi' \\ &= (a+b)^2 - (a-b)^2\sin^2\phi' \end{aligned}$$

Preform another trignometric substitution and collect powers of $\sin\phi'$,

$$\begin{aligned} (1-\sin^2\phi')((a+b)^2 - (a-b)^2\sin^2\phi') + 4b^2\sin^2\phi' &= (a+b)^2 \\ &\quad +(4b^2 - (a-b)^2 - (a+b)^2)\sin^2\phi' \\ &\quad +(a-b)^2\sin^4\phi' \end{aligned}$$

Since $4b^2 - (a-b)^2 - (a+b)^2 = -2(a+b)(a-b)$,

$$\begin{aligned} \Gamma &= (a+b)^2 - 2(a+b)(a-b)\sin^2\phi' + (a-b)^2\sin^4\phi' \\ &= ((a+b) - (a-b)\sin^2\phi')^2 \end{aligned}$$

Lemma 3

$$(a^2\cos^2\phi + b^2\sin^2\phi)^{-1/2}d\phi = (a'^2\cos^2\phi' + b'^2\sin^2\phi')^{-1/2}d\phi'$$

Proof: The differential of,

$$\sin\phi = \frac{2a\sin\phi'}{a+b+(a-b)\sin^2\phi'}$$

is,

$$\cos\phi\;d\phi = \frac{2a\cos\phi'\Delta - 2a\sin\phi'(a-b)2\sin\phi'\cos\phi'}{\Delta^2}d\phi'$$

where $\Delta = a+b+(a-b)\sin^2\phi'$. So,

$$\begin{aligned} \Delta^2\cos\phi\;d\phi &= 2a\cos\phi'(\Delta - 2(a-b)\sin^2\phi') \\ &= 2a\cos\phi'\,(a+b-(a-b)\sin^2\phi')\;d\phi' \end{aligned}$$

Substituting for $(\Delta\cos\phi)$ and canceling $2\cos\phi'$,

$$\Delta(a'^1\cos^2\phi' + b'^2\sin^2\phi')^{1/2}\;d\phi = a(a+b-(a-b)\sin^2\phi)\;d\phi'$$

Then apply the previous lemma.

References

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