AGM calculation of Pi

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Highlights

Historic:

In 1799, Gauss was startled to discover that his *arithmetic-geometric mean* connected ϖ, the half-circumference of a curve known as the lemniscate, with π, the half-circumference of a unit circle:

> That the AGM is equal to π/ϖ between 1 and $\sqrt{2}$ we have confirmed up to the 11-th decimal digit; if this is proven, then a truly new field of analysis stands before us.

Gauss also found efficient ways of computing many elliptic integrals, including those for ϖ, thus giving an extremely fast algorithm for the computation of π.

In 1976 the method was rediscovered independently by Brent and Salamin and stands currently as among the fasted known methods for calculating π.

Mathematical:

The half-circumference ϖ of the lemniscate curve is given by an *elliptic integral of the first kind.* This integral can be related to π using another elliptic integral, one said to be of *the second kind.* Elliptic integrals can be numerically evaluated with the help of an iterative process known as the arithmetic-geometric mean. Combining these integrals it is possible to express π in a manner suitable for efficient numerical evaluation.

Equations

$$\begin{aligned} x_1 &\rightarrow (x_1 + x_2)/2 \\ x_2 &\rightarrow \sqrt{x_1 x_2} \end{aligned}$$

$$\begin{array}{rcl} x_3 & \to & x_3 - x_4(x_1 - x_2)^2/4 \\ x_4 & \to & 2\,x_4 \\ x_5 & \to & (x_1 + x_2)^2/(2\,x_3) \end{array}$$

Variables

$$\begin{array}{lcl} x_1 & : & \text{the arithmetic mean} \\ x_2 & : & \text{the geometric mean} \\ x_3 & : & \text{the arc length of a lemniscate} \\ x_4 & : & 2^k \\ x_5 & : & \text{the } k\text{-th convergent to } \pi \end{array}$$

Dynamics

Discussion

Gauss's fast method for the computation for the circumference of a circle begins with investigations into the circumference of another plane curve, the *lemniscate*. In polar coordinate, the leminscate is given by,

$$r^2 = \cos 2\theta.$$

This curve was intriguing to Gauss and others because it had mathematical properties related to the circle. The half circumference of the leminscate is $\varpi = 2\,I(\sqrt{2}, 1)$, where $I(a, b)$ is the *elliptic integral of the first kind*,

$$I(a, b) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2 \cos^2\theta + b^2 \sin^2\theta}}.$$

In fact, the symbol ϖ for the half-circumference of a "unit radius" leminscate is a variant of π, the symbol for the half-circumference of a "unit radius" circle.

Although elliptic integrals are exceeding common and important in practice, they are impossible to evaluate analytically. Gauss showed that these integrals can be numerically evaluated by successively replacing a and b by their arithmetic and geometric means,

$$\begin{array}{rcl} a_{i+1} & = & (a_i + b_i)/2 \\ b_{i+1} & = & \sqrt{a_i b_i} \end{array}$$

where $a_0 = a$ and $b_0 = b$. The sequence a_i and b_i converge extremely quickly to a common value, called the *arithmetic-geometric mean of a and b*, denoted $\mathrm{AGM}(a, b)$. Specifically, in 1799 Gauss discovered that if a_i and b_i follow from a and b as a result of one or several steps of the AGM procedure, the integral is unchanged, $I(a, b) = I(a_i, b_i)$. Hence, in the limit, the common value $\mathrm{AGM}(a, b)$ can be factored out and the integrand collapses to $1/\sqrt{\cos^2\theta + \sin^2\theta} = 1$,

$$I(a, b) = \frac{\pi}{2\,\mathrm{AGM}(a, b)},$$

and in particular,

$$\pi/\varpi = \mathrm{AGM}(\sqrt{2}, 1).$$

Gauss found this result astounding. A qualitative relationship between the theory of leminscates and circles had been noted, and now it was found that the ratio of the fundamental constants in the two theories are related by a value proceeding from the process of arithmetic-geometric mean!

An additional relation is required in order to use the AGM to calculate π, rather than the ratio π/ϖ. Among the various elliptic integrals under study at that time, there were those of the second kind,

$$L(a, b) = \int_0^{\pi/2} \frac{\cos^2\phi\, d\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}}.$$

Gauss also gave a method to numerically evalute $L(a, b)$ using the AGM,

$$c_0^2 L(a, b) = (c_0^2 - S)I(a, b),$$

where,

$$S = \sum_{k=0}^{\infty} 2^{k-1} c_k^2,$$

$c_i^2 = a_i^2 - b_i^2$, and the a_i and b_i are the iterates in the AGM process of a and b.

In 1748, Euler discovered that,

$$L(\sqrt{2}, 1)I(\sqrt{2}, 1) = \pi/4$$

Substituting this into the relations above, Gauss derived,

$$\pi = \frac{\mathrm{AGM}(\sqrt{2}, 1)^2}{1 - S}.$$

The use of this formula is that the AGM converges extremely quickly, and the values c_i, required in the evaluation of the sum S, converge towards zero extremely quickly. With just several terms we already have a very good approximation of π.

A map suitable for Phaser is derived from this formula. Let x_1 and $_2$ be a_i and b_i, x_3 be $1-S$, the sum S truncated to the first k terms, and x_4 be 2^k. At each iteration,

1. x_1 is updated to be the arithmetic mean of x_1 and x_2;
2. x_2 is updated to be the geometric mean of x_1 and x_2;
3. x_3 is updated to include an additional term;
4. x_4 is updated to $2x_4 = 2^{k+1}$.

The update of x_3 requires values for c_k^2 and 2^k. Note that c_i^2 can be written as $(a_{i-1} - b_{i-1})^2/4$, which reduces the amount of computation needed to derive this value. The value of 2^k is available in x_4.

Let x_5 be the approximation to π given by the current approximation to $\mathrm{AGM}(1, \sqrt{2})$ and S. At each iteration we compute this from x_1, x_2 and x_3. It is customary to take as the approximate value of the AGM at step i to be the arithmetic mean of a_i and b_i.

References

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