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Competitive Auctions and Digital Goods

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Abstract

We study a class of single round, sealed bid auctions for items in unlimited supply such as digital goods. We focus on auctions that are truthful and competitive. Truthful auctions encourage bidders to bid their utility; competitive auctions yield revenue within a constant factor of the revenue for optimal fixed pricing. We show that for any truthful auction, even a multi-price auction, the expected revenue does not exceed that for optimal fixed pricing. We also give a bound on how far the revenue for optimal fixed pricing can be from the total market utility. We show that several randomized auctions are truthful and competitive under certain assumptions, and that no truthful deterministic auction is competitive. We present simulation results which confirm that our auctions compare favorably to fixed pricing. Some of our results extend to bounded supply markets, for which we also get truthful and competitive auctions.

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1 Introduction

Consider the problem of selling a number of identical items to consumers who each want a single item and the items are available in *unlimited* supply. By unlimited supply we mean that either the seller has at least as many items as there are consumers, or that the seller can reproduce items on demand at negligible marginal cost. Of particular interest are digital items such as downloadable audio files and pay-per-view movies. With unlimited supply, consumer *utilities*, the maximum amounts that consumers are willing to pay for the item, are the sole factor determining sale prices. The seller's goal is to maximize their total revenue.

One way to set prices for items in unlimited supply is to estimate consumer utility via market analysis and then set a fixed price. We refer to this method as *fixed pricing*. Pay-per-view movies are an example of fixed pricing for an unlimited supply market. With perfect knowledge of consumer utilities, *optimal fixed pricing* maximizes fixed-price revenue by selecting the optimal price at which to sell items. Fixed pricing generally cannot achieve this ideal due to the inherent inaccuracy of market analysis. If the price is set too high, not enough items may be sold; if the price is set too low, insufficient revenue may be collected per item.

Auctions automatically adjust prices to market conditions. We study single round, sealed bid auctions. Such auctions have been studied for items available in *scarce supply*, where maximizing the revenue requires that all available items be sold. They are especially practical when the number of consumers is large. In particular, Vickrey [18] introduced a multi-item auction that is *truthful*. A truthful auction encourages bidders to bid their utility value. In an untruthful auction, bidders may bid significantly below their utility values, reducing auction revenue.

In a truthful auction, rational bidders bid their utilities. In addition, we would like such an auction to be *competitive*: it must yield revenue within a constant factor of optimal fixed pricing. To be competitive, a truthful auction must vary how many items are sold depending on the bid values. For example, as we show in Section 3, the multi-item Vickrey auction is not competitive if the seller chooses the number of items to sell before knowing the bid values (and not truthful if the seller chooses the number of items after knowing the bid values). As with fixed pricing, selling too few or too many items may not maximize revenue. Thus, the method for choosing how many bids to satisfy is an integral part of a truthful competitive auction.

To our knowledge, auctions have never been studied in a competitive framework. Nor are any existing auctions competitive in the sense that we introduce here. As we explore in Sections 11 and 12, this competitive framework is useful in studying any kind of auction where identical goods are being sold, not just auctions for unlimited supply.

Auctions are becoming a popular pricing mechanism in electronic commerce, both for human users and for trading agents (bots). In many cases, the use of auctions is complicated by the fact that a good bidding strategy for a buyer requires an understanding of strategies and utilities of other buyers. Truthful auction mechanisms may be attractive in this context because they avoid this complication.

In this paper we study a class of truthful auctions for unlimited supply. We study both *single-price* auctions, where every winning bidder pays the same price, and *multi-price* auctions, where the prices may differ. In addition to *deterministic* auctions, we study *randomized* auctions that use randomization to decide which bids to fill and at what price. We develop techniques for design and analysis of auctions for unlimited supply. Our approach is reminiscent of competitive analysis of on-line algorithms [1, 16], where performance of an on-line algorithm is gauged in terms of performance of an optimal off-line algorithm. Here the optimal off-line algorithm is analogous to the optimal fixed pricing mechanism.

Although we develop our results for unlimited supply, some of the results extend to *bounded supply*, where the number of items for sale is bounded, but maximizing revenue might not result in all items being sold. We discuss bounded supply in Section 11.

We view auctions as algorithms for deciding which input bids to fill, and at what price. As with any algorithm, one needs to address the issues of correctness, efficiency, and performance. In the context of this paper, an auction is correct if the auction is truthful and fills each winning bid at or below the bid value. Efficiency of an auction refers to the time needed to process bids. The auctions introduced in this paper are very fast; sorting of the input bids is the most expensive computational operation we perform. As discussed above, we measure auction performance by its revenue relative to the fixed pricing revenue. This computer science approach allows us not only to design new truthful auctions, but also to give theoretical guarantees for their performance. Such provable performance guarantees are new to the area of auction mechanism design.

To state our results more formally, we introduce the following notation. Let n denote the number of bidders. Without loss of generality, we assume that the lowest bid is one and denote the highest bid by h. Let the total utility $\mathcal{T}$ be the sum of all bidders' utilities. $\mathcal{T}$ is an obvious upper bound on the revenue that can be obtained from this set of bidders. Let $\mathcal{F}$ be the optimal fixed pricing revenue. Clearly $\mathcal{F} \leq \mathcal{T}$. $\mathcal{F}$ is an upper bound on the revenue that can be obtained by any fixed priced sale or any single-price auction. We want revenues of our truthful auctions to be competitive with $\mathcal{F}$. We assume that h is small compared to $\mathcal{F}$.[1] This assumption prevents a trivial upper bound on the revenue; see Section 2.

We state some of our results in terms of the total utility, $\mathcal{T}$, and others in terms of the optimal fixed pricing revenue, $\mathcal{F}$. We show that $\mathcal{F}$ compares favorably to $\mathcal{T}$; specifically, $\mathcal{F} = \Omega(\mathcal{T}/\log h)$, and also $\mathcal{F} = \Omega(\mathcal{T}/\log n)$. This result shows that the optimal fixed pricing revenue is within a $\min(\log h, \log n)$ factor of the revenue of any pricing scheme. We use this result to relate various bounds expressed in terms of $\mathcal{T}$ to those expressed in terms of $\mathcal{F}$.

We introduce a class of truthful single-price auctions and a class of truthful multi-price auctions. A randomized auction from the first class is competitive: it has an expected revenue of $\Omega(\mathcal{F}) = \Omega(\mathcal{T}/\log h)$. A dual-price variant of this auction has revenue that is close to $\mathcal{F}$ if $\mathcal{F}/h$ is large enough. A randomized multi-price auction from the second class has an expected revenue of $\Omega(\mathcal{T}/\log h)$. Thus, both of these auctions have the same worst-case bound in terms of $\mathcal{T}$. However, we show that the latter auction is not competitive. Its expected revenue is $\Omega(\mathcal{F}/\sqrt{\log h})$, and this bound is tight: on certain inputs, the expected revenue is $O(\mathcal{F}/\sqrt{\log h})$.

This provides support for using $\mathcal{F}$, rather than $\mathcal{T}$, to define competitive auctions. Another result provides further support. We show that for any truthful auction, even a multi-price one, the expected revenue does not exceed $\mathcal{F}$. This result is somewhat surprising: using single-price auctions does not hurt revenue by more that a constant factor.

A natural question to ask is if there is a truthful deterministic auction with an $\Omega(\mathcal{F})$ revenue. We show that there is none: for any such auction, there is a set of bids that leads to an $O(\mathcal{F}/h)$ revenue, i.e., revenue that is a small fraction of $\mathcal{F}$ if h is large. Thus, for worst-case performance, randomized auctions yield better revenue than the deterministic ones.

Our theoretical analyses are limited in that their performance metrics are accurate only up to a constant factor. As a result, the analyses do not reveal whether one of our auctions dominates the others, or which auction is better for a natural distribution. As a supplement to our theoretical results, we performed a number of simulations to compare our auctions with each other and to fixed pricing on a variety of input families. Our simulations suggest that, on natural inputs, some of our auctions attain revenue very close to $\mathcal{F}$ if the number of bids is large enough. Furthermore, our auctions can outperform fixed pricing with market analysis unless that analysis is fairly accurate. We also show a deterministic auction that, despite the worst-case result, does very well on natural inputs.

We develop a framework for a theoretical and experimental analysis of revenue-maximizing truthful auctions and introduce auctions that perform well in this framework. We show how algorithm analysis techniques can be used within this framework to obtain results that are interesting and in some cases surprising.

2 Competitive Analysis

We consider auctions with n bidders, each bidder i having a utility value u_i and bidding b_i. We also assume that the bids are ordered so that $b_i \leq b_{i+1}$. In auctions where ties need to be broken, we can assume an arbitrary total order on the bid values that respects the partial order. That is, we can assume that the order given by the indices is strict. We assume that there is no collusion among the bidders.

Given a set of bids, the outcome of an auction is the subset of bids that are satisfied and a corresponding set of sale prices such that, for each winning bid b_i, the associated sale price is at most b_i. A *deterministic auction mechanism* maps sets of bids to auction outcomes. A *randomized auction mechanism* maps

[1] Some of our results hold under weaker assumptions.

sets of bids to probability distributions on auction outcomes. We use $\mathcal{R}$ to denote the auction *revenue* for a particular auction mechanism and set of bids. $\mathcal{R}$ is the sum of all sale prices. For randomized auctions, $\mathcal{R}$ is a random variable.

We say that an auction is *truthful* if bidding u_i is a dominant strategy for bidder i. More specifically, let a *bidder's profit* be the difference between the bidder's utility value and the price the bidder pays if they win the auction, or zero if they lose. Then an auction is truthful if a bidder's profit (or expected profit, for randomized auctions), as a function of the bidder's bid, is maximized at the bidder's utility value, for any fixed values of the other bidders' bids. Truthful auctions encourage bidding at utility value if the bid value that maximizes the (expected) profit is unique. If the bid value is not unique, truthfulness at least does not discourage bidders from bidding their utility value. When considering truthful auctions, we assume that $b_i = u_i$.

To enable analysis of auction revenue we define several properties of an input set of bids. As stated in the introduction, $\mathcal{T}$ is the sum of all of the bids. An equivalent definition of $\mathcal{T}$ is the revenue due to the optimal multi-price untruthful auction, the one that satisfies all bids at their value. The revenue for optimal fixed pricing is $\mathcal{F}$. Note that $\mathcal{F}$ can also be interpreted as the revenue due to the optimal single-price untruthful auction. More discussion of $\mathcal{F}$ and its relation to the optimal single-price untruthful auction is given in Section 4. Other properties that we use in analysis are ℓ, the smallest bid value, and h, the highest bid value. Because bids can be arbitrarily scaled, we assume, without loss of generality, that $\ell = 1$, in which case h is really the ratio of the highest bid to the lowest bid.

Analogous to on-line algorithm theory, we express auction performance relative to that for optimal untruthful auctions, as ratios $\mathcal{R}/\mathcal{T}$ or $\mathcal{R}/\mathcal{F}$. However, we solve a maximization problem while on-line algorithms solve minimization problems. Thus, positive results, which are lower bounds on $\mathcal{R}/\mathcal{T}$ or $\mathcal{R}/\mathcal{F}$, are expressed using "Ω". Impossibility results, which are upper bounds on $\mathcal{R}/\mathcal{T}$ or $\mathcal{R}/\mathcal{F}$ for any auction in a certain class, are expressed using "O".

Note that h, $\mathcal{F}$, and $\mathcal{T}$ are used only for analysis. Our auctions work without knowing their values in advance.

If we do not impose any restrictions on h, we get the upper bound of $\mathcal{R}/\mathcal{T} = O(1/h)$. To see this, imagine $n-1$ bids at 1 and one bid, b_n, at h. An auction that wishes to do better than $O(1/h)$ must base the selling price on bidder n's bid. However, this would encourage bidder n to bid below u_n.

To prevent this upper bound on auction revenue we can make the assumption that the optimal revenue $\mathcal{F}$ is significantly larger than h, the highest bid. That is, for some constant α, we assume that $\alpha h \leq \mathcal{F}$. With this assumption, optimal fixed pricing sells at least α items. In some cases α is a fixed constant. In other cases, success probability approaches 1 as $\alpha \to \infty$. For some proofs, this assumption is stronger than what we need and we make the weaker assumption that $\alpha h \leq \mathcal{T}$.

We say that an auction is *competitive* under certain assumptions if when the assumptions hold, the auction's revenue is $\Omega(\mathcal{F})$, or equivalently $\mathcal{R}/\mathcal{F} = \Omega(1)$.

3 Prior Results

Our results are related to the field of mechanism design that combines microeconomic motivation with game-theoretic tools and includes auction mechanisms. For introduction to the area, see for example [10, 13]. In particular, auctions for scarce supply markets have been extensively studied. See [14] for a survey. Some work in the Computer Science community combines economic or game-theoretic questions with computational questions. Earlier results are surveyed in [9]; for more recent results, see e.g. [4, 8, 12].

Truthful auctions are an example of strategyproof mechanisms. Such mechanisms have been developed for several goals. For example, the Vickrey–Clarke–Groves mechanism [3, 6, 18] maximizes the general welfare of a system. The Shapley Value [15] mechanism shares costs among the participants. We address a less well-understood problem of maximizing the revenue of one of the parties (the seller).

The k-item Vickrey auction [18] was a starting place for our work. The k-item Vickrey auction is a single-price auction that sells k items to the k highest bidders at the price equal to the $k+1$ highest bid (b_{n-k-1}). For generalizations of Vickrey auctions to the multiple resource case, see e.g. [3, 6, 17].

In the unlimited supply case, taking $k = f(n)$ yields a truthful auction for any function f with $1 \leq f(n) < n$. This auction mechanism is not competitive, however. To see this, consider a *bipolar* input that has k bids at h and $n-k$ bids at 1. In this case $\mathcal{R} = k$ and $\mathcal{F} \geq hk$. This gives an $\mathcal{R}/\mathcal{F} = O(1/h)$ bound. As we show later no deterministic auction can do much better on worst-case distributions, but a randomized auction can.

4 Optimal Untruthful Auctions

In this section we study the two untruthful auctions, the optimal multi-price auction and the optimal single-price auction, and establish the relationship between their revenues, $\mathcal{T}$ and $\mathcal{F}$. We show that $\mathcal{F} \geq \mathcal{T}/(2\log h)$. This bounds the penalty for requiring auctions to be single-price and allows us to compare bounds expressed in terms of $\mathcal{T}$ with those expressed in terms of $\mathcal{F}$.

To get a better understanding of how the optimal single-price auction works, we define the *optimal threshold function*, $\mathrm{opt}(B)$. This function on a set of bids B returns the fixed price at which items should be sold to achieve revenue $\mathcal{F}$. In the optimal single-price auction, all bid values that are at least $\mathrm{opt}(B)$ will be satisfied at price $\mathrm{opt}(B)$. All other bids will be rejected. That is,

$$\mathrm{opt}(B) = \mathrm{argmax}_{b_i \in B}\, b_i \cdot (n - i + 1).$$

Note that $n-i+1$ is the number of bids that are at least b_i. The main result of this section is as follows.

Theorem 4.1 $\mathcal{F} \geq \mathcal{T}/(2\log h)$.

Proof. Divide the bids into $\log h$ bins by partitioning the bids at values of powers of two. Thus, each bid is less than a factor of two from any other bid in the same bin. Since the sum of the bids is $\mathcal{T}$ and there are $\log h$ bins, then some bin has a sum of at least $\mathcal{T}/\log h$. Note that the lowest bid in this bin is at least half of any other bid in the bin. If the optimal single-price auction chose, as its selling price, the price of the lowest bid in this bin, then the contribution of each bid in this bin to the revenue is at least half of the bid's value. Since the bin sums to more than $\mathcal{T}/\log h$, this means that the revenue is greater than $\mathcal{T}/(2\log h)$. Thus the optimal fixed pricing can always achieve a revenue of at least $\mathcal{T}/(2\log h)$. ■

One can make this bound strongly polynomial as suggested by Satish Rao and Eva Tardos.

Corollary 4.2 $\mathcal{F} \geq \mathcal{T}/(4\log n)$.

Proof. Let p be the optimal price; clearly $p \geq h/n$. If one drops all bids with values below p, $\mathcal{F}$ does not change and $\mathcal{T}$ decreases by at most a factor of two. After the bids are dropped, the ratio of the highest and the lowest bid values is at most n, yielding the desired result. ■

Now we turn our attention to truthful auctions.

5 Generalized Truthful Auction Mechanisms

By making observations about auctions that encourage utility value bids, in particular the Vickrey auction, we can design general auction mechanisms. We present two general auction mechanisms that facilitate the design of truthful auctions. The first, the *bid-independent auction mechanism*, is based on the observation that the price that a bid is satisfied at must be independent from that bid's actual value. The second, the *random sampling auction mechanism*, is based on the observation that rejected bids can be used to set prices for bids that are to be satisfied.

5.1 Bid-Independent Auctions

The first general truthful auction mechanism that we discuss is one that is typically multi-price, although some variants are single-price. The motivation for this mechanism is the observation that bidder i's bid value should only determine whether bidder i wins or loses the auction (as a threshold). The bid value should not determine bidder i's price. As we will show in section 9.3, for deterministic auctions turthfulness is equivalent to being bid-independent.

Let B be the set of all bids and let B_i be the set of bids without bidder i's bid. A bid-independent auction uses a predetermined function, f, from sets of bid values to prices. The auction works as follows. Bidder i wins the auction at price $f(B_i)$ if $b_i \geq f(B_i)$. [2] Otherwise, the bidder loses.

[2] For maximum generality, we allow f to also return whether to use $\geq$ or $>$ in this comparison. For the perpose of maximizing revenue we can allways just use $\geq$; however, to prove Lemma 9.2 we need the full generality.

Lemma 9.2 shows that this auction mechanism is quite general: any truthful deterministic auction is equivalent is bid-independent.

We can immediately see how this generalizes the traditional Vickrey auction. The 1-item Vickrey auction fits into this general framework with $f = \max$. Note that bidder n wins this auction and pays b_{n-1}. If f is the function that returns the kth highest element of the set of bids, we get the k-item Vickrey auction.

5.2 Random Sampling Auctions

Another general truthful auction mechanism is based on randomized sampling. We select a subset B' of B at random, independent of the bid values. Let m be the size of B'. We then compute a function on these sampled bids, $f(B')$, and use this as a threshold value for the $n-m$ bids in the non-sample, $B \setminus B'$. Note that this auction mechanism is inherently single-price.

If a multi-price auction is acceptable, then this auction can be modified to be dual-price with $m \approx n/2$, by using $f(B')$ to compute the threshold to use for bids in $B \setminus B'$ and $f(B \setminus B')$ as the threshold to use for bids in B'. This is a good way to avoid revenue loss due to the rejected bids in the sample; however, it is at the expense of making the auction dual-price.

6 Random Sampling Optimal Threshold Auction

The random sampling optimal threshold auction takes $f = \text{opt}$, the optimal threshold function, in the random sampling auction. Intuitively we use the sample to get an idea for a good threshold value, then we apply that threshold to the remaining bids. The auction samples $m = n/2$ bids at random, computes opt of this sample, and uses this value as a threshold for the non-sample, accepting all bids above this threshold at the threshold value. In this section we show that the expected revenue of this auction is within a constant factor of $\mathcal{F}$, assuming $\mathcal{F}/h$ is not too small.

For the purpose of simplifying our analysis, we will be analyzing a different method of sampling, one that selects a bid to be in the sample independently at random with probability 1/2. This method of sampling is simpler to analyze, and it does worse than the former (this can easily be seen when the probabilistic bounds are discussed).

In practice, for the single-price auction, we might want to set $m = n/10$ or even $m = \sqrt{n}$. For the dual-price version of the auction, $m = n/2$ is a good choice.

6.1 Performance Analysis

In this section we show that, under certain assumptions, the expected revenue of the random sampling optimal threshold auction is within a constant factor of $\mathcal{F}$. This result implies that restricting a single-price auction to be truthful does not affect performance by more that a constant factor. As we have seen, restricting a multi-price auction to be truthful may affect its performance by roughly a factor of $\log h$.

Our analysis of the random sampling auction uses the following lemma, which is a variation of the Chernoff bound (see e.g. [2, 11]).

Lemma 6.1 *Consider a set A and its subset $B \subset A$. Suppose we pick an integer k such that $0 < k < |A|$ and a random subset (sample) $S \subset A$ of size k. Then for $0 < \delta \leq 1$ we have*

$$\mathbf{Pr}[|S \cap B| < (1-\delta)|B| \cdot k/|A|] < \exp(-|B| \cdot k\delta^2/(2|A|)).$$

Proof. We refer to elements of A as points. Note that $|S \cap B|$ is the number of sample points in B, and its expected value is $|B| \cdot k/|A|$. Let $p = k/|A|$. If instead of selecting a sample of size exactly k we choose each point to be in the sample independently with probability p then the Chernoff bound would yield the lemma.

Let $A = \{a_1, \ldots, a_n\}$ and without loss of generality assume that $B = \{a_1, \ldots, a_k\}$. We can view the process of selecting S as follows. Consider the elements of A in the order induced by the indices. For each element a_i considered, select the element with probability p_i, where p_i depends on the selections made up to this point.

Let t be the number of points already selected when a_{i+1} is considered. Then $i-t$ is the number of points considered but not selected. Suppose that $t/i < p$. Then $p_{i+1} > p$.

We conclude that when we select the sample as a random subset of size k, the probability that the number of sample points in B is less than the expected value is smaller than in the case we select each point to be in the sample with probability p. ∎

Let $\mathcal{R}$ be the revenue of the random sampling optimal threshold auction. The following theorem shows that $\mathcal{R} = \Omega(\mathcal{F})$ with probability going to one as α goes to ∞.

Theorem 6.2 *Assume $\alpha h \leq \mathcal{F}$. Then $\mathcal{R} \geq \mathcal{F}/6$ with probability of at least $1 - e^{-\alpha/36} - 40e^{-\alpha/72}$.*

Proof. Let k be the number of bids satisfied in the optimal single-price solution. Consider the optimal fixed pricing revenue of the sample, $\mathcal{F}'$. Since we expect $k/2$ of these bids to be in the sample, $\mathbf{E}[\mathcal{F}'] \geq \mathcal{F}/2$. Applying Lemma 6.1 with A the set of all bids, B the set of bids in the optimal threshold solution on A, and $\delta = 1/3$, we conclude that $|S \cap B| < |B|/3 = k/3$ with probability at least $1 - e^{-k/36}$. The assumption $\alpha h \leq \mathcal{F}$ implies $k \geq \alpha$. Thus $\mathcal{F}' \geq \mathcal{F}/3$ with probability at least $1 - e^{-\alpha/36}$.

Let k_s be the number of bids satisfied by optimal fixed pricing on the sample (i.e. the number of bids in the sample that are at least $\text{opt}(B')$) and let k_n be the number of bids in the non-sample that are at least $\text{opt}(B')$. If $\mathcal{F}' \geq \mathcal{F}/3$ then $k_s \geq \alpha/3$.

Now, assuming that $k_s \geq \alpha/3$, we show that the probability that $k_n < k_s/2$ is small. Note that $k_n < k_s/2$ implies that among the top $(3/2)k_s$ bids, at least k_s are in the sample. Note that the sample and the non-sample are symmetric: taking a random subset containing half of the elements in A is equivalent to taking a complement of a random subset of half of the elements. We apply Lemma 6.1 with S being the non-sample, B being the top $i = (3/2)k_s$ bids, and $\delta = 1/3$, and conclude that the probability that $k_n < k_s/2 = i/3$ is at most $e^{-i/36}$.

If $\mathcal{R} < \mathcal{F}'/2$ then $k_n < k_s/2$ and thus for some $i \geq (3/2)\alpha/3 = \alpha/2$ we have $k_n < i/3$. Using the union bound, the probability that this happens is at most

$$\sum_{i=\alpha/2}^{\infty} e^{-i/36} < 40e^{-\alpha/72}.$$

Using the union bound for the probabilities that $\mathcal{R} \geq \mathcal{F}'/2$ and $\mathcal{F}' \geq \mathcal{F}/3$, we conclude that $\mathcal{R} \geq \mathcal{F}/6$ with probability at least $1 - e^{-\alpha/36} - 40e^{-\alpha/72}$. ∎

Note that in the above theorem, we can trade off the bound on $\mathcal{R}$ and the probability that this bound holds. In particular, for any constant $\epsilon > 0$, we can show that $\mathcal{R} \geq \mathcal{F}/(2+\epsilon)$ with probability that goes to 1 as α goes to infinity, but the convergence is slower for smaller ϵ.

By symmetry, the expected revenue of the dual-price variant of the random sampling auction is twice the expected revenue of the original. One can show that the expected revenue of the dual-price auction is at least $\mathcal{F}/(1+\epsilon)$ if α is large enough.

On the other hand, for may input distributions the threshold value of a sample has a non-zero probability of being non-optimal for the non-sample. If this is the case, the expected value of the revenue in the random sampling auction that samples half of the bids is less than $\mathcal{F}/2$, and the expected value of the dual price auction is less than $\mathcal{F}$.

7 Weighted Pairing

All truthful auctions we introduced so far are either single-price or dual-price. In this section we describe a multi-price auction. The *weighted pairing* auction we present is in the bid-independent class with the function f defined as follows:

$$f(B) = b \in B \text{ w.p. } \tfrac{b}{\sum_{b' \in B} b'}$$

Thus, to determine if bidder i wins the auction and at what price, pick a bid $b \in B_i$ with probability proportional to the value of b, i.e., $b/(\mathcal{T} - b_i)$. This pairs b_i with b. If $b \leq b_i$, bidder i wins at cost b, otherwise i loses. Note that the result of this selection for i does not affect the auction outcome for the bidder who bid b.

To understand the intuition behind this auction, consider a related *random pairing* auction. Assume that n is even and pair bidders at random, independent of their bid values. For each pair, conduct a 1-item Vickrey auction, that is, for a pair (b_i, b_j) with $b_i < b_j$, i loses and j wins at cost b_i.

Compared to the random pairing auction, a bidder in the weighted pairing auction is less likely to win, but when a bidder wins they are likely to pay more. It turns out that high bidders are still likely to win the auction, and the benefit of high bidders paying higher prices outweighs the benefit of more low bidders winning the auction. In particular, Theorems 7.1 and 8.2 imply that for the weighed pairing auction, in the worst case $\mathbf{E}[\mathcal{R}]$ is proportional to $\mathcal{T}/\log h$ in worst case. For the (unweighted) pairing auction, we have shown that the worst case $\mathbf{E}[\mathcal{R}]$ is proportional to $\mathcal{T}/\sqrt{h}$. We do not include the latter result because it is dominated by the former.

Next we prove the following result.

Theorem 7.1 *If $4h \leq \mathcal{T}$, then for the weighted pairing auction $\mathbf{E}[\mathcal{R}] = \Omega(\mathcal{T}/\log h)$.*

Proof. Partition the n bids into $\log h$ bins as before such that bin j contains only bids in the interval $[2^{j-1}, 2^j]$. Recall that an important property of these bins is that bids in the same bin are within factor of two from each other. Let S_j be the sum of the bids in bin j. Note that the sum of the contents of bins that contain only one bid is upper bounded by $\sum_{j=1}^{\log h} 2^j = 2h - 1 < \mathcal{T}/2$. We will ignore such bins in our analysis below. Consider all bins with two or more bids, and let $\mathcal{T}'$ be the sum of bids in these bins. We have $\mathcal{T}' > \mathcal{T}/2$.

For each bin j, we look at pairings of bids that are both in j and we bound the expected revenue due to such pairings. First, the probability that a bid i in bin j is paired with another bid in bin j is $(S_j - b_i)/(\mathcal{T} - b_i) > S_j/(3\mathcal{T})$, since bin j contains at least two bids and $S_j - b_i \geq S_j/3$.

Let $b'_1, \ldots, b'_k$ be the values of bids in bin j in the increasing order. Given that a bid b'_i from the bin is paired with another bid in the same bin, the probability that the bid wins is at least

$$\frac{i-1}{2(k-i)+(i-1)} > \frac{i-1}{2k}.$$

This comes from assuming that all bids below bid i are at value 2^{j-1} and all bids above bid i are at value 2^j. This is the worst it could possibly be. The expected number of bids in bin j that win when paired with other bids from the bin is at least

$$\sum_{i=1}^{k} \frac{i-1}{2k} = \frac{k-1}{4}.$$

The smallest bid in bin j has the value of at least $S_j/(2k)$. Let $\mathcal{R}_j$ be the revenue generated by bids in bin j being paired with other bids in bin j. We have:

$$\mathbf{E}[\mathcal{R}_j] \geq \frac{S_j}{3\mathcal{T}} \cdot \frac{S_j}{2k} \cdot \frac{k-1}{4} \geq \frac{(k-1)S_j^2}{24k\mathcal{T}}$$

Since $k \geq 2$, we have $(k-1)/k \geq 1/2$ so

$$\mathbf{E}[\mathcal{R}_j] \geq \frac{S_j^2}{48\mathcal{T}}$$

We are interested in $\mathbf{E}[\mathcal{R}] \geq \sum \mathbf{E}[\mathcal{R}_j] \geq \sum \frac{S_j^2}{48\mathcal{T}}$. Since S_j's sum up to $\mathcal{T}'$, the sum is minimized when all $\log h$ of the S_j's are equal to $\mathcal{T}'/\log h$. In this case,

$$\mathbf{E}[\mathcal{R}] \geq \frac{\mathcal{T}'^2(\log h)}{48\mathcal{T}\log^2 h}$$

Since $\mathcal{T}' > \mathcal{T}/2$, we have

$$\mathbf{E}[\mathcal{R}] \geq \frac{\mathcal{T}}{192\log h}$$

Thus, $\mathbf{E}[\mathcal{R}] = \Omega(\mathcal{T}/\log h)$. ■

Note that the constant here is 1/192 which does not seem too good. However, the analysis was very loose in contributions to $\mathcal{R}$ that it considered. In practice, this auction has much better constants. See Section 10 for details.

Lower bounds on revenues of the weighted pairing and the random sampling optimal threshold auctions, stated in terms of $\mathcal{T}$, are the same, $\Omega(\mathcal{T}/\log h)$. The next result shows that the latter auction has a better worst-case performance.

We show that the expected revenue of the weighted pairing auction is $\Omega(\mathcal{F}/\sqrt{\log h})$, and that this bound is tight in the worst case. This implies that the wighted pairing auction is not competitive.

We use the following notation. We denote the expected revenue of the wighted pairing auction by W, let $k = \log h$ and $s = \sqrt{\log h}$ First we give bound the revenue from below.

Theorem 7.2 *If $\mathcal{F} \geq 2h$, then $W = \Omega(\mathcal{F}/\sqrt{\log h})$ and this bound is tight.*

Proof. If $\mathcal{F} \leq \mathcal{T}/s$, then since $W = \Omega(\mathcal{T}/k)$ we have $W = \Omega(\mathcal{T}/s)$.

Assume $\mathcal{F} > \mathcal{T}/s$. Partition the bids over buckets, with bucket i containing bids in the range $[2^i, 2^{i+1})$. Let M be the set of bids in a bucket with the largest total bid value and assume that M is defined by the value range $[t, 2t)$. We show that the revenue due only to M is big. The assumption $\mathcal{F} \geq 2h$ implies that $|M| \geq 2$. (If $|M| = 1$ than $M = \{h\}$ and the previous bucket contains a single bid; in this case $\mathcal{F} < 2h$.) Recall that $|M|t = \Omega(\mathcal{F})$. Let M' be the highest $\lfloor |M|/2 \rfloor$ elements of M and let $M'' = M \backslash M'$.

The probability that an item in M' is paired up with an item in M'' is $\Omega(\mathcal{T}/\mathcal{F}) = \Omega(1/s)$, and the expected revenue is $\Omega(|M'|t/s) = \Omega(\mathcal{F}/s)$.

The above argument considers only contributions of items in M. Somewhat surprisingly, the resulting bound is tight up to a constant factor. We give an example where the expected revenue W of the weighted pairing auction is $O(\mathcal{T}/s)$. The bid values are 2^i for $i = 1, \ldots, k$. For each $i = 1, \ldots, k-1$ the number of bids of this value is 2^{k-i} and the total value of such bids is 2^k. For the last value, 2^k, there are s bids of this value for the total of $s2^k = \mathcal{F}$. Note that $\mathcal{T} = 2^k(s + k - 1)$.

We show that $W = O(2^k)$ by first showing that the expected contribution of all bids of value less then 2^k to W is $O(2^k)$ and then showing that the contribution of all bids of value 2^k is $O(2^k)$.

For $i < k$, the probability that a bid of value 2^i wins is less than

$$\frac{i2^k - 2^i}{T - 2^i} \leq \frac{i2^k}{T} \leq \frac{i}{k}.$$

The expected contribution of a winning bid is less than

$$\frac{1}{i}\sum_{j=1}^{i} 2^j < \frac{1}{i}2^{i+1}.$$

This all bids of value 2^i contribute less than $2^{k+1}/k$. Summing over $i = 1, \ldots, k-1$, we conclude that the expected contribution of the corresponding bids is less than $2 \cdot 2^k$.

Next we consider the bids of value 2^k. The probability that such a bid wins is at least one and the expected revenue of a winning bid is at least

$$\frac{s}{k+s}2^k + \frac{1}{k+s}\sum_{i=1}^{k-1} 2^i < \frac{s+1}{k+s}2^k = \frac{2^k}{s}.$$

Thus the total contribution of such bids is less than 2^k. ∎

Although the weighted pairing auction is not competitive in the worst case, it is only a factor of $\sqrt{\log h}$ away from being competitive, is quite different from our random sampling auctions, and performs relatively well on inputs that are bad for the latter. One may be able to improve this auction; see Section 13.

8 An Upper Bound

A natural question to ask is if the bound of Theorem 7.1 can be improved upon. We already know that no single-price auction can do better than $\mathcal{F}$. In this section we prove that no truthful multi-price auction can have an expected revenue greater than $\mathcal{F}$. This result contrasts that of Section 6. In that section, we show that a truthful single-price auction performance is within a constant factor of the optimal single-price auction. Results of this section imply that no truthful multi-price auction performance is within a constant factor of the optimal multi-price auction.

Consider a collection of bids $B = \{b_1, \ldots, b_n\}$ with $b_i \leq b_{i+1}$. Note that $b_1 = 1$ and $b_n = h$. Define the following quantities, which are dependent on the auction mechanism:

p_i	the probability a bid i is satisfied,
c_i	expected cost to winning bidder i,
g_i	expected profit (gain) for bidder i.

With probability p_i a bidder i wins. The bidder's expected gain, having won, is their utility value minus the expected price they paid, i.e.

$$g_i = p_i(u_i - c_i) \tag{1}$$

Now we show that in a truthful auction, probabilities of winning are monotone functions of bid values.

Lemma 8.1 *Suppose in a truthful auction $b_i < b_j$. Then $p_i \leq p_j$.*

Proof. Suppose that i and j have the same utility equal to b_i. Since the auction is truthful, i's gain is at least as big as j's, thus

$$p_i(b_i - c_i) \geq p_j(b_i - c_j).$$

Similarly, if both utilities are b_j, we get

$$p_j(b_j - c_j) \geq p_i(b_j - c_i).$$

Adding the above inequalities and simplifying, we get

$$p_i b_i + p_j b_j \geq p_j b_i + p_i b_j.$$

Rearranging, we get

$$p_j(b_j - b_i) \geq p_i(b_j - b_i).$$

Since $b_j > b_i$, we have $p_j \geq p_i$. ∎

The main result of this section is as follows.

Theorem 8.2 *For any truthful auction,* $\mathbf{E}[\mathcal{R}] \leq \mathcal{F}$.

Proof. In a truthful auction, if a bidder $i-1$ had the utility value of b_i but bids b_{i-1}, their gain would not exceed g_i, thus

$$g_i \geq p_{i-1}(b_i - c_{i-1}). \tag{2}$$

So,

$$\begin{aligned} g_i &\geq p_{i-1}(b_i - b_{i-1} + b_{i-1} - c_{i-1}) \\ &= p_{i-1}(b_i - b_{i-1}) + p_{i-1}(b_{i-1} - c_{i-1}) \\ &= p_{i-1}(b_i - b_{i-1}) + g_{i-1}. \end{aligned}$$

We can recursively expand g_{i-1} in the same way until we get to g_1 which is 0 because all bids are satisfied at value at least 1, and get

$$g_i \geq \sum_{j=1}^{i-1} p_j(b_{j+1} - b_j). \tag{3}$$

Now let $\mathcal{R}_i$ be the total expected revenue from bidder i. That is

$$\mathcal{R}_i = p_i c_i.$$

We can rearrange equation (1) as $p_i c_i = p_i b_i - g_i$ and get

$$\mathcal{R}_i = p_i b_i - g_i.$$

Using equation (3) we get

$$\mathcal{R}_i \leq p_i b_i - \sum_{j=1}^{i-1} p_j(b_{j+1} - b_j).$$

Looking at the sum of the $\mathcal{R}_i$'s, we see that the first term is mostly canceled by the summation term and we get a telescoping effect.

$$\begin{aligned} \mathbf{E}[\mathcal{R}] &= \sum_{i=1}^{n} \mathcal{R}_i \\ &\leq \sum_{i=1}^{n} p_i b_i - \sum_{i=1}^{n} \left[\sum_{j=1}^{i-1} p_j(b_{j+1} - b_j) \right]. \end{aligned}$$

By counting the number of times each $p_j(b_{j+1} - b_j)$ occurs, we can rearrange the second summation to get

$$\mathbf{E}[\mathcal{R}] \leq \sum_{i=1}^{n} p_i b_i - \sum_{j=1}^{n-1} p_j(b_{j+1} - b_j)(n-j)$$

$$\begin{aligned} \mathbf{E}[\mathcal{R}] &\leq p_n b_n + \sum_{j=1}^{n-1} p_i b_i - \sum_{j=1}^{n-1} p_j(b_{j+1} - b_j)(n-j). \\ &= p_n b_n + \sum_{j=1}^{n-1} p_j \left[b_j - (b_{j+1} - b_j)(n-j) \right]. \end{aligned}$$

Regrouping

$$\mathbf{E}[\mathcal{R}] \leq p_n b_n + \sum_{j=1}^{n-1} p_j \left[b_j(n-j+1) - b_{j+1}(n-j) \right].$$

Now, let $V_j = b_j(n-j+1)$. Intuitively, this is the revenue attained by using b_j as the sale price in fixed pricing. Note that $V_j \leq \mathcal{F}$.

$$\mathbf{E}[\mathcal{R}] \leq p_n V_n + \sum_{j=1}^{n-1} p_j \left(V_j - V_{j+1} \right).$$

Rearrange this sum to sum over V_j instead of p_j and for symmetry, define $p_0 = 0$.

$$\mathbf{E}[\mathcal{R}] \leq \sum_{j=1}^{n} \left(p_j - p_{j-1} \right) V_j.$$

But, $V_j \leq \mathcal{F}$ and by Lemma 8.1, $p_j - p_{j-1}$ is non-negative.

$$\mathbf{E}[\mathcal{R}] \leq \mathcal{F} \sum_{j=1}^{n} (p_j - p_{j-1})$$

This sum telescopes to $p_n - p_0$, but since $p_0 = 0$, we have

$$\mathbf{E}[\mathcal{R}] \leq p_n \mathcal{F} \leq \mathcal{F}.$$

■

9 Deterministic Auctions

We have shown several randomized auctions with an $\Omega(\mathcal{F})$ expected performance under the assumption that $\alpha h \leq \mathcal{F}$ and α is big enough. In this section we study deterministic auctions.

First we study the bid-independent version of the optimal threshold auction, a natural auction that performs well on some input distributions. We show that its worst case performance, however, is $O(\mathcal{F}/h)$.

We extend this upper bound by showing that any truthful deterministic bid-independent auction has an $O(\mathcal{F}/h)$ worst-case performance even for $\alpha h \leq \mathcal{F}$. Then we show that any deterministic truthful auction is bid-independent. This implies an upper bound on worst-case performance for all truthful deterministic auctions.

9.1 Deterministic Optimal Threshold Auction

The *deterministic optimal threshold auction* is the bid-independent auction with $f = \text{opt}$, the optimal threshold function defined in Section 4. The only difference between the deterministic optimal threshold auction and the optimal fixed pricing mechanism is that the former uses threshold $\text{opt}(B_i)$ for bidder i and the latter uses $\text{opt}(B)$. Recall that B_i and B only differ in that b_i is not in B_i. From this, we might also expect that for large n with suitable constraints on h, the the deterministic optimal threshold auction would perform to within a constant fraction of $\mathcal{F}$. However, this is not the case as the following example illustrates.

The input that shows that the deterministic optimal threshold auction performance is $O(\mathcal{F}/h)$ is one with r bids at value h and $q = (h-1)r - 1$ bids at value 1. On this input, the optimal single-price auction takes r high bids at price h for a revenue of $\mathcal{F} = hr$. Note that the second-best threshold would take all n bids at price 1 for revenue of $hr - 1$. The deterministic optimal threshold auction takes all the high bids at value 1 and none of the low bids. This yields a revenue of $\mathcal{R} = r$. None of the low bids are accepted because their threshold is still h (removing a low bid from the optimal threshold auction does not change the threshold). The high bids are taken at value 1 because removing one h causes the optimal threshold to switch to 1. Thus, the deterministic optimal threshold auction on this input family has revenue of $\mathcal{R} = \mathcal{F}/h$.

Next we generalize this result.

9.2 Upper Bound for Deterministic Bid-Independent Auctions

In this section we prove the following upper bound.

Theorem 9.1 *For any truthful deterministic bid-independent auction and any constant α, there exists an input for which $\mathcal{R}/\mathcal{F} = O(1/h)$ and $\alpha h \leq \mathcal{F}$.*

Proof. Let f be the (deterministic) function that defines the auction. Consider an input with n_h bids at value h and n_ℓ bids at value 1. Restricted to such inputs, f is a function of n_h and n_ℓ. Note that we can assume, without loss of generality, that f takes on only two values: 1, in which case all bids are satisfied at the price of 1, and h, in which case the high bids are satisfied at price h. Other values of f lead to smaller revenues.

We wish to find an input family such that the bid-independent auction with function f has revenue $O(\mathcal{F}/h)$. To obtain this family, we chose n_h and n_ℓ in such a way that $f(n_h - 1, n_\ell) = 1$, $f(n_h, n_\ell - 1) = h$, and $n_h > \alpha$ (implying $\alpha h \leq \mathcal{F}$). For such an input, $\mathcal{R} = n_h$ and $\mathcal{F} \geq h n_h$ so $\mathcal{R}/\mathcal{F} \leq 1/h$. Our goal now is, given a deterministic f, to find values of n_h and n_ℓ that have the above properties.

Consider the n_h, n_ℓ plane. For a fixed m look at the line $n_h = k$ and $n_\ell = m - k$, and consider the line segment connecting $(0, m)$ and $(m, 0)$. We need to find a value of k with $k > \alpha$ where, when k increases by one, f changes from 1 to h.

Set $m = h^2\alpha$. Assume that $f(\alpha, m - \alpha) = 1$. As we increase k from $k = \alpha$ the value of $f(k, m - k)$ must change from 1 to h because for $k = m$ we have $f(k, m - k) = f(m, 0) = h$. Thus it must be at some k^* that $f(k^*, m - k^*) = h$ and $f(k^* - 1, m - k^* + 1) = 1$. If we now choose $n_h = k^*$ and $n_\ell = m - k^* + 1$, we satisfy our criteria that $n_h > \alpha$ and that $f(n_h - 1, n_\ell) = 1$ and $f(n_h, n_\ell - 1) = h$. Thus, $\mathcal{R}/\mathcal{F} = O(1/h)$.

Suppose now that our assumption that $f(\alpha, m - \alpha) = 1$ is false and instead it is h. Then for $n_h = k = \alpha$ and $n_\ell = m - k + 1$ we have $\mathcal{F} = m + 1 = h^2\alpha + 1$

and $\mathcal{R} \leq h\alpha$ so

$$\begin{aligned}\mathcal{R}/\mathcal{F} &\leq \tfrac{h\alpha}{h^2\alpha+1} \\ &\leq \tfrac{h\alpha}{h^2\alpha} \\ &= \tfrac{1}{h}\end{aligned}$$

Thus for any bid-independent auction with deterministic function f there exists an input family such that $\mathcal{R} = O(\mathcal{F}/h)$. ∎

9.3 Truthful Deterministic Auctions are Bid-Indepentent

The following lemma allows us to extend the result of Theorem 9.1 to arbitrary deterministic auctions.

Lemma 9.2 *Any truthful deterministic auction is bid-independent.*

Proof. Let $B_i = B \setminus \{i\}$ as before, and let B_i^x be the set of bids with b_i replaced with value x. Let $\mathcal{A}$ be a truthful deterministic auction. Let $\mathcal{A}(B)$ denote the result of running $\mathcal{A}$ on set B and let $\mathcal{A}_i(B)$ denote the result for bidder i (∞ if i is rejected). Consider function g defined as $g(x) = \mathcal{A}_i(B_i^x)$. Here g is clearly a function of $\mathcal{A}$ and B_i. We show that g is ∞ everywhere except for some region (v, ∞), $[v, \infty)$, or $\{\}$ where it has value v. This would imply that $\mathcal{A}$ is bid-independent. If $g(x) = \infty$ for all x then we have the empty interval, $\{\}$. Otherwise, let $b = \inf\{x : g(x) \neq \infty\}$ and let $v = \min_x g(x)$. We now show that $b = v$ and for all $b' > b$, $g(b') = v$.

1. $v = b$

   First we show that $v \leq b$. This is simple to see because $g(x)$ is defined so that either $g(x)$ is ∞ (and a bid value of x will lose) or $g(x) \leq x$ (and a bid of value x will win at price $g(x)$ which is necessarily at most x).

   Now we show that $v \geq b$. Assume for a contradiction that there exists a b' such that $v < b' < b$. This auction is not truthful because a bidder with utility value b' would be better off bidding greater than b, win the aucion and pay price v which is less than b'. Thus, $v = b$.

2. For all $b' > b$, $g(b') = v$.

   Let b'' be such that $g(b'') = v$. Since $b' > v$, if $g(b')$ were not exactly v a bidder with utility value b' would be better off bidding b'' and paying only v. Since the auction is truthful, it must be that $g(b') = v$.

The only loose end remaining is showing that $g(b)$ is v or ∞. This is true because $b = v$ and by definition either $g(b) \leq b$ or $g(b) = \infty$.

∎

Using Lemma 9.2, we generalize Theorem 9.1 as follows.

Theorem 9.3 *For any truthful deterministic auction and any constant α, there exists an input for which $\mathcal{R}/\mathcal{F} = O(1/h)$ and $\alpha h \leq \mathcal{F}$.*

In terms of asymptotic worst-case performance, deterministic auctions are significantly worse than randomized auctions. This is not to say that deterministic auctions are bad to use for all input families. In fact our experimental results reveal that for many families, the deterministic optimal threshold auction works very well. Adequate knowledge of the bidding distribution may make it possible to use a deterministic auction.

10 Experimental Results

Our theoretical analysis of auctions has limitations. The worst-case analysis leaves a $\sqrt{\log h}$ gap for inputs which come from "typical" (as opposed to tailored to be hard) distributions. In addition, constant factors we obtain in our analysis are often too pessimistic. Theoretical analysis for specific distributions seems non-trivial even for simple distributions.

In practice, constant factors of the auction revenue are important. We introduced several auction mechanisms that provably perform within a constant factor of each other in worst-case. However, we do not know which one is better. We would also like to know how these auctions compare to fixed pricing with imperfect market analysis.

We turn to experiments to answer these questions. In our experiments, we simulate various auctions on several input families and see how they compare.

10.1 Experimental Setup

Our experiments consist of picking a family of bidder utilities with a single parameter and a set of auctions and comparing resulting revenues for different parameter values. If either the family is randomized or the auction mechanism is randomized then the revenue for the auction is average over a number of runs. The number of times we repeat the auction depends on the size of the set of bids. This is because standard deviations are bigger for small sets of bids.

10.2 Auction Mechanisms

We experimented with the following auctions: the deterministic optimal threshold, the weighted pairing, and variations on the randomized sampling optimal threshold. We have limited our presentation of the experimental results to the following specific auctions:

DSO	dual-price sampling optimal threshold.
SSO	single-price sampling optimal threshold, $m = \sqrt{n}$. (We justify this choice of m later.)
WP	weighted pairing.
DOT	deterministic optimal threshold.
FP−	fixed pricing with the price equal to optimal − 25%.
FP+	fixed pricing with the price equal to optimal + 25%.

The revenue for fixed pricing is at most $\mathcal{F}$, which can be achieved by an optimal price choice. If the bid distribution is known in advance and well-behaved, one can choose a near-optimal price. However, if the distribution is not known or changes, the selling price might not be near optimal. FP− and FP+ simulate fixed pricing with non-optimal selling price by setting the selling price to the optimal price minus and plus 25%, respectively. Note that the former's revenue is at least 75% of $\mathcal{F}$ while for the latter the revenue can be zero, for example if all bids are the same.

The auctions we introduce adapt to the input bid distribution. Comparing their revenue to that of FP− and FP+ allows us to see how well our auctions adapt.

10.3 Input Families

We found the following input families to be of interest because of their average-case or worst-case properties.

uniform(low,high) Each bid is chosen independently from the uniform distribution distributed between *high* and *low*. Note that the ratio of *high* to *low* is an upper bound for h. In our experiments we tried various moderate and large values for this ratio as well as the extreme value for $low = 0$ and $high = 1$. The results were similar in all cases, so we present data for the latter distribution only. Note that for this distribution, $\mathbf{E}[\ell] = 1/(n+1)$ and $\mathbf{E}[h] = n/(n+1)$.

normal(mean,dev) Each bid is chosen independently from the normal distribution with mean *mean* and standard deviation *dev*. We have actually skewed the normal distribution so as not to allow negative bid values. If a negative bid is generated we discard it and pick another bid from the normal distribution to replace it.

Zipf(theta,high) Each bid is chosen independently from the Zipf distribution [19]. This is a generalization of the distribution with 80% of the total bid value coming form 20% of the bidders. For i in the interval $[1, \textit{high}]$, we define $\mathbf{Pr}[X = i] = c/i^{\theta}$, with c chosen so that the probabilities integrate to one. We tried several values of θ, and report the results for $\theta = 1/2$; according to G. K. Zipf, this distribution models personal income.

equal-revenue(α) Inputs in this family have $h = n/\alpha$. These inputs have the property that for any bid except for the largest α bids, setting the selling price to that bid value and satisfying all bids greater than or equal to this value yields the same revenue. Thus, bid $b_i = n/(n-i)$ if $n - i > \alpha$ and n/α otherwise. This distribution is bad for several auctions.

bipolar(low,high,ratio) This is the bipolar family with bids at *high* or at *low* only. The ratio of the number of high bids to the total number of bids is *ratio*. In the experiment we report on, we keep high, low, and the total number of bids constant, and vary the ratio.

10.4 Size Simulations

For the all problem families we report on, except for the bipolar family, we ran a size test where we varied the number of bidders between 10 and 100,000. The auctions tested behaved roughly in the same manner on the uniform, normal, and Zipf families. These "average-case" families have the property that any

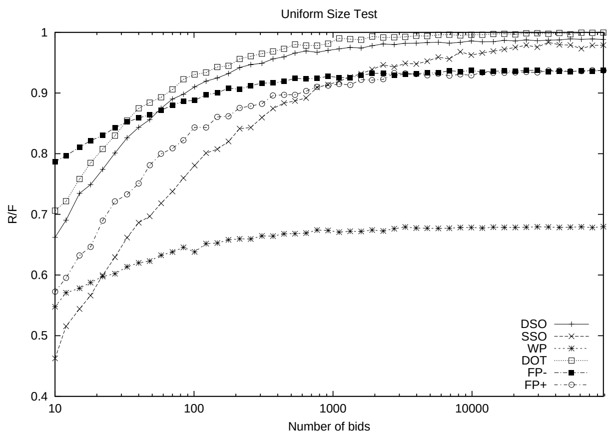


Figure 1: Uniform(0,1) with $n = [10, 100k]$

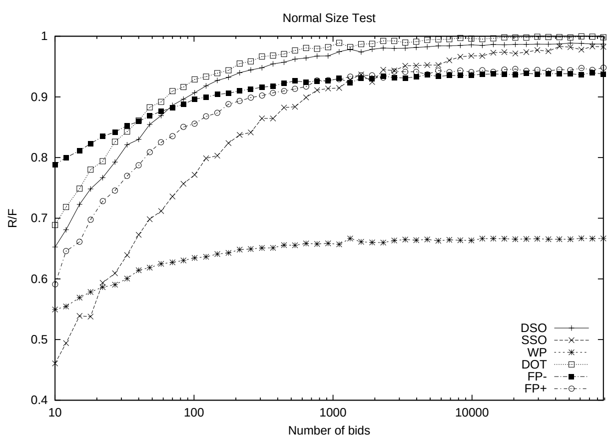


Figure 2: Normal(1,1) with $n = [10, 100k]$

uniformly chosen random subset of the bids has the same distribution as the original. In particular, the random sample in a random sampling auction has the same distribution as the non-sample. Because of this property, the random sampling auctions perform very well these families.

Figures 1, 2, and 3 show the results of simulation. For large n, DSO and DOT are the best auctions. As n increases, the ratio of their revenue to $\mathcal{F}$ approaches 1. This is also the case for SSO. As n increases, the ratios for FP$-$ and FP$+$ approach a constant less than 1. As a result, even for moderately large n our best auctions perform better than fixed pricing with a 25% price error. On average-case distributions, WP is the worst auction; its ratio

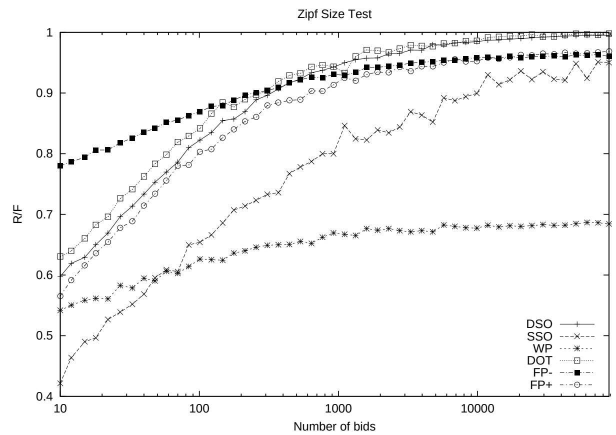


Figure 3: Zipf(theta=0.5, high=n) with $n = [10, 100k]$

asymptotically approaches 2/3.

Equal-revenue is a "worst-case" family for many auctions, as we see in Figures 4 and 5. Since this family is such that the fixed pricing with any price between 1 and h works well, we do not plot FP$-$ and FP$+$.

WP, the worst auction on the average-case families, is the best of our auctions on the equal-revenue families. On these families, the sampling optimal threshold mechanisms with larger sample sizes perform poorly in comparison to the ones with smaller sample sizes. This is why DSO performs worse than SSO (More experiments with sample size in the sampling optimal threshold mechanism are presented later). DOT performs poorly on the equal-revenue families because it satisfies only the highest bids at the price of the lowest bid.

Recall that our theoretical bounds improve as α increases. Comparing the equal-revenue distributions with α equal to 1.0 (the lowest possible value for any input) and 10.0, we observe that auction performance is better for the higher value of α. However, even for $\alpha = 1.0$ (in which case most of our theoretical lower bounds do not apply), our randomized auctions bring a revenue that is a substantial fraction of $\mathcal{F}$.

10.5 Bipolar Family

For the bipolar family, we varied the ratio of high bids to the total number of bids and computed the revenue for various auctions. The results appear in

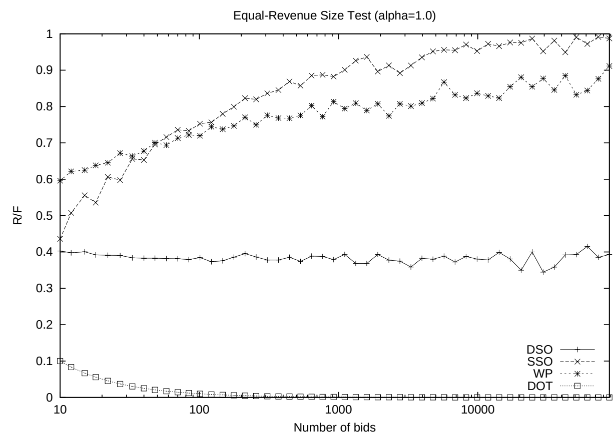


Figure 4: Equal-Revenue($\alpha = 1.0$) with $n = [10, 100k]$

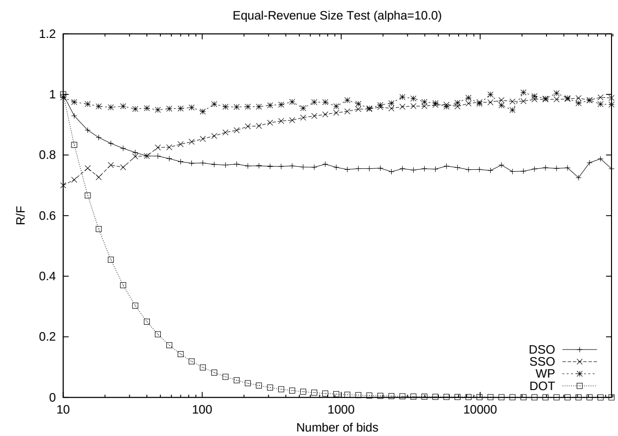


Figure 5: Equal-Revenue($\alpha = 10.0$) with $n = [10, 100k]$

Figure 6. We do not plot FP− and FP+ auction revenues.

There are several key things to note about the bipolar family. First, it demonstrates the problem with the DOT auction. There is a sharp dip in the revenue of DOT precisely when the number of high bids is 10 and the number of low bids is 90. This is because $\text{opt}(B \setminus \{h\}) = 1$ and $\text{opt}(B \setminus \{1\}) = h$ in this scenario and thus 10 high bids get satisfied at price 1 and the 90 low bids get rejected[3]. The optimal solution in this case is to accept the 10 high bids at price $h = 10$. Aside from this case, DOT performs well. The randomized optimal threshold variants have degraded revenues for the number of high bids around 10. However, due to randomness, the revenue loss is significantly smaller, but spread over a wider region. The DSO auction outperforms WP and SSO for most ratios. The WP auction usually outperforms SSO.

Note that WP revenue actually spikes up in the area where the optimal threshold auctions' revenues dip. This is more evidence to the notion that the WP seems to do well on worst-case families.

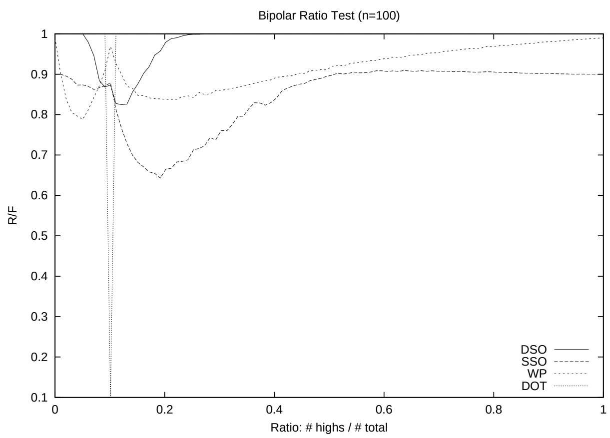
[3]Note that in this scenario 1 and h produce the same revenue when used as a threshold for $B \setminus \{h\}$; however, f must deterministically break the tie one way or the other. We have chosen to break ties in favor of using a smaller threshold. Note that if we had made the opposite choice then we would have encountered the worst-case behavior on the distribution with with 9 bids at h and 91 bids at 1 which has the property that 1 and h yield equal revenues when used on $B \setminus \{1\}$.

10.6 Sample Size

In this section we address the choice of the sample size for the sampling optimal threshold auction. Figures 7 and 8 show the revenue of the single-price sampling optimal threshold auction as the sample size varies. One plot is for $n = 100$ and the other for $n = 10,000$. We plot a curve for every input distribution except for the equal-revenue with $\alpha = 10.0$ (which is similar to that with $\alpha = 1.0$) and the bipolar distribution. We omit the latter because otherwise we need to plot a curve for several values of the high to low bid ratio.

These plots show that for the average-case distributions, there is a tradeoff: a bigger sample allows to choose a better threshold, but loses revenue due to the discarded sample. For the equal-revenue distribution, there is no tradeoff; smaller samples lead to higher revenues as any threshold is a good threshold. The sample size of around $\sqrt{n}$ (10 for the first plot and 100 for the second) is a good compromise, giving near-maximum values for all our distributions.

These plots help explain why DSO performs so poorly on the equal-revenue distribution. The best sample size for this distribution is one. If the value of the single sampled bid is b, the auction revenue is $\mathcal{F} - b$, and the expected value of b is small. If we sample more than one element then the we are likely to select a threshold value that has more than its fair share of elements above it in the sample – and less than its fair share of elements above that value in



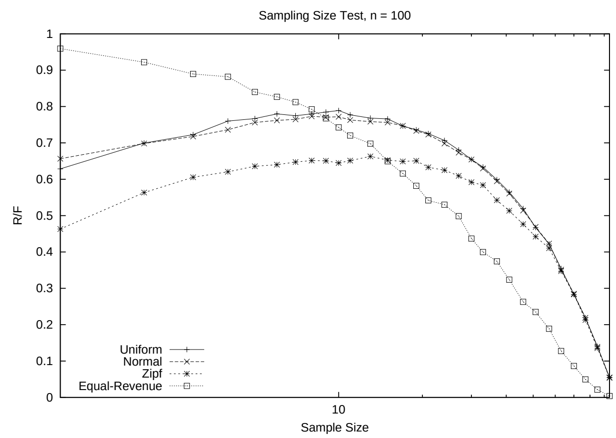


Figure 6: Bipolar(low=1,high=10,ratio=$[0,1]$) with $n = 100$

Figure 7: The effect of sample size on the sampling optimal threshold auction revenue.

the non-sample. In this sense, the sampling optimal threshold auction mechanisms are poorly conditioned on the equal-revenue distribution.

One question related to the sample size is whether the dual-price auction can do better with lopsided sample sizes. It does do slightly better in auctions with a small number of bids, but as the number of bids gets large the benefit of accepting bids from the sample becomes negligible. Note that a dual-price auction with the sample size $\sqrt{n}$ will do better than the single-price sampling optimal threshold auction with the same sample size.

10.7 Experimental Conclusions

Our experiments suggest that for well-behaved inputs, the random sampling auctions achieve a revenue ratio to $\mathcal{F}$ that approaches one as the number of bids increases. The weighted pairing auction, although not bad overall, does not perform as well as the randomized sampling mechanisms on such inputs. We also saw that even on contrived worst-case families, these auctions' revenue is a large constant fraction of $\mathcal{F}$. The deterministic optimal threshold, while exhibiting its $\mathcal{F}/h$ worst-case behavior on specific families, performs very well on the average-case families of inputs. Finally, sampling about a square root of the number of bids for the sampling auction mechanisms seems to balance out the loss due to rejecting all of the sample and the loss due to having a non-representative sample.

11 Bounded Supply

Up to this point, we have studied the unlimited supply case which is motivated by the digital goods market where the cost of making a copy of an item is negligible. In this section we consider the case where the number of items available for sale is bounded. This case is typical of physical goods markets. We denote the number of items available by k. Here again, the seller wishes to maximize their revenue, possibly not selling all of the items. Note that the definitions of truthful and competitive auctions, which we stated for the unlimited supply case, also apply to the bounded supply case. We denote by $\mathcal{F}_k$ the revenue for optimal fixed pricing that sells at most k items, and it is this quantity that we wish to be competitive with.

The bounded supply case is a generalization of the unlimited supply case as items are available in unlimited supply when the number of available items is the same as the number of bidders (i.e. $k = n$). Where unlimited supply is one extreme of the bounded supply case, *scarce* supply is another extreme. In the scarce supply case, the optimal fixed pricing revenue is maximized by selling all the items when the number of available items is around k.[4] Previous work on auctions concentrated on the scarce supply case. The multi-item Vickrey auction is the best single-price

[4] Formally, items are in scarce supply if we have k items and the optimal fixed pricing with k or $k+1$ items would sell all of the items.

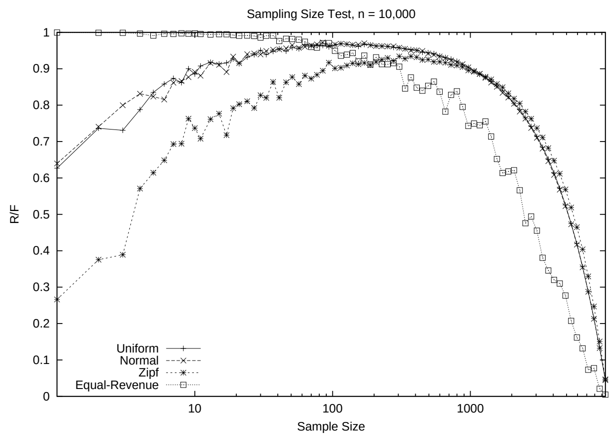


Figure 8: The effect of sample size on the sampling optimal threshold auction revenue.

auction for the scarce supply case (and it is competitive). Our results, extended to bounded supply, are competitive in the full range of the supply-demand spectrum with only the assumption that $\alpha h < \mathcal{F}_k$.

We now show how to extend our optimal threshold sampling auctions to the k-item bounded supply case. Let opt_k be the function that, given a set of bids, returns the optimal threshold that sells k items or less. This function, on the entire set of bids, gives the threshold to use for $\mathcal{F}_k$. The single-price sampling optimal threshold mechanisms can now be modified to use threshold function $\mathrm{opt}_{mk/(n-m)}$ on sample of size m. If this results in too many bids being satisfied, arbitrarily (e.g. at random) reject bids until there are only k left. One can show that with high probability, the number of bids rejected this way will be small and that this auction is competitive.

In the dual-price auction with the sample size of $m = n/2$, use $\mathrm{opt}_{k/2}$ so that about $k/2$ bids are selected from each of the sample and the non-sample. Once can show that the resulting auction is also truthful and competitive.

We have shown that in a relatively straight-forward way, our sampling auctions extend from the unlimited to the bounded supply case. One can extend the deterministic optimal threshold auction to bounded supply as well. Also, all of our upper bounds apply to bounded supply because it generalizes unlimited supply.

To generalize Theorem 4.1 for the k-item case, recall the alternative definition of $\mathcal{T}$ as the revenue due to the optimal untruthful multi-price auction. Define $\mathcal{T}_k$ to be the revenue of the optimal multi-price auction restricted to only satisfying k bidders (i.e. the sum of the highest k bids) and $\mathcal{F}_k$ as above. Then the generalized result is $\mathcal{F}_k \geq \mathcal{T}_k/(2\log h)$.

One result we do not know how to extend to bounded supply is that for the weighted pairing auction.

12 Concluding Remarks

We have demonstrated that there exist truthful auctions for unlimited supply markets. We have shown randomized auctions that are competitive in that they yield revenue that is within a constant factor of optimal fixed pricing. We have shown that this result is tight up to a constant factor, even for multi-price auctions. We have also shown that no deterministic auction is competitive in the worst-case. Finally, via simulations, we have argued that our auctions compare favorably to fixed pricing with market analysis.

For unlimited supply markets, our analysis assumes that there is no cost for producing the items being auctioned. With the following modification we can also accommodate non-zero marginal costs. If the marginal cost is v per item, then first subtract v from each bid and reject all negative value bids. After running the auction, add v back to the selling price of all winning bids. If the marginal cost of producing k items is a more complicated function of k, we can modify the opt function used in the sampling optimal threshold auction to take into account these marginal costs. In this case the opt function would, as above for bounded supply, need to be parameterized by the ratio of the sample size and the non-sample size so as to correctly use marginal cost information.

To prevent cheating by the auctioneer or the bidders, one may need a trusted third party or a special cryptographic protocol. Note that cheating prevention is a problem shared by all on-line auctions. Related results appear in [5].

Recently we have shown that the deterministic optimal threshold auction is single-price. This fact leads to interesting results. We will report on these results in a future paper.

13 Open Problems

We proved that the weighted pairing auction is not competitive. It is possible, however, that a variant of this auction that uses different weighting is competitive.

The weighted pairing auction can have revenue greater than $\mathcal{F}$ for some random pairings. It seems like this might not be the case for the dual-price sampling optimal threshold auction. Can one prove that it is always the case that for any randomly chosen sample, the auction revenue does not exceed $\mathcal{F}$?

We have not considered issues such as the extent to which bidders can remain anonymous or bid values can remain secret. See [7] for ways to maintain bid secrecy in an on-line Vickrey auction.

A significant issue in auctions like ours is resistance to adversarial attacks. How resistant are our auctions to bidder collusion, and can collusion resistance be improved? How well do our auctions resist attacks such as a competitor attempting to reduce the revenue of an auction by submitting a large number of low bids?

Repeated auctions for the same item may be of interest in some applications. In this case, the challenge is to design an auction mechanism that encourages consumers to bid their utilities in every auction (I.e. they are truthful).

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