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Solution Set 4

1. Problem 3.2.2: If $P_e$ is a while-program computing $\Phi : N^2 \rightarrow N$, show $P_e(e,a) = P_e(a,0)$ for all $a$.

If $P_e(e,a)$ halts then $\Phi(e,a)$ is defined and therefore $P_e(a)$ halts. Conversely, if $P_e(a)$ halts, then $\Phi(e,a)$ is defined so $P_e(e,a)$ halts. Therefore:

$$ P_e(e,a) = \Phi(e,a) = P_e(a), $$

where equality is understood in the extended sense where, in addition to the standard equality of naturals, “non-halting” equals “not-defined”.

By Definition 6, Section 2.3, $P_e(a) = P_e(a,0)$. This completes the proof.

2. Problem 3.2.3: Let $f : N \rightarrow N$ be a computable bijection. Enumerate all while-programs by:

$$ P_{f(0)}, P_{f(1)}, \ldots. $$

Prove the existence of a universal function $\Psi$ for this enumeration and show $\Psi = P_{f(n)}$ for some $n$.

The principal result of Chapter 3 is that the universal function $\Phi(x,y)$ is computable. But then so is,

$$ \Psi(a,b) = \Phi(f(a),b), $$

which is the universal function for the new enumeration. So $\Psi(a,b)$ is given by some arity two program $P_m$. The surjectivity of $f$ assures the existence of an $n$ such that $m = f(n)$. Therefore, the statement remains true if $f$ is total and onto but not if it is total and one-to-one but not onto.

3. Problem 3.2.4: Let $\Phi : N^2 \rightarrow N$ be the universal function. Show that $\theta(x) = \Phi(x,x)$ cannot be extended to a total computable function.

Following the hint, let $\theta' \geq \theta$ be total and computable. Let

$$ \theta''(x) = \theta'(x) + 1. $$
It is clear that $\theta''$ is computable if $\theta'$ is, so $\theta'' = P_j$. So,

$$\theta'(j) + 1 = \theta''(j) = P_j(j) = \Phi(j,j) = \theta(j) = \theta'(j).$$

Note that each of these equalities is a true equality: on the assumption that $\theta'$ is total, each function is defined at $j$ and the $j$-th while-program halts. We use this in the assertion $\theta'(j) = \theta(j)$ which is only true if $\theta$ is defined at $j$. But $\theta'(j) = \theta'(j) + 1$ is a contradiction.

In order to understand this proof better. Suppose $\theta' \geq \theta$ was a computable function but perhaps not total. We still can define a computable $\theta''$ as before and the index $j$ still exists. The equation,

$$\theta'(j) + 1 = \theta'(j),$$

which has no solutions over the naturals, does have a solution if we allow $\theta'(j) = \bot$ — a non-terminating computation followed by an application of the successor function is still a non-terminating computation. Therefore $\theta'$ is not total.

4. Problem 3.2.5: Show the existence of a function $\phi_i$ for which no algorithm can decide if $\phi_i(x)$ is defined for an arbitrary $x \in N$.

Let $\phi_i(x) = \Phi(x,x)$. This function is computable. Note that the question whether $\phi_i(x)$ is defined on $x$ is equivalent to whether the $x$-th while-program halts on $x$, which is not computable.