Burton Rosenberg
Test 1 Answers

The four problems were graded as follows:

1. 5 points.

2. (a) 2 points.
   (b) 1 point.
   (c) 2 points.

3. (a) 2 points for $L_i$, 2 points with for $P_i$.
   (b) 1 point.

4. $\Rightarrow$ 3 points.
   $\Leftarrow$ 2 points.
1. [Simplex Method]

Solve the Following LP showing step-by-step the simplex method:

\[
\begin{align*}
\text{max} & \quad x_1 + 2x_2 + x_3 \\
\text{s.t.} & \quad x_1 + x_2 + x_3 \leq 2 \\
& \quad x_1 + x_2 \leq 1 \\
& \quad x_1, x_2, x_3 \geq 0
\end{align*}
\]

Introduce slack variables:

\[
\begin{align*}
x_4 &= 2 - x_1 + x_2 + x_3 \\
x_5 &= 1 - x_1 - x_2 \\
z &= x_1 + 2x_2 + x_3
\end{align*}
\]

The initial basis is \(\{x_4 = 2, x_5 = 1\}\). We pivot \(x_2\) into the basis, and \(x_5\) out:

\[
\begin{align*}
x_2 &= 1 - x_1 - x + 3 \\
x_4 &= 1 - x_3 + x_5 \\
z &= 2 - x_1 + x_3 - 2x_5
\end{align*}
\]

Now pivot \(x_3\) in and \(x_4\) out:

\[
\begin{align*}
x_2 &= 1 - x_1 - x_5 \\
x_3 &= 1 - x_4 + x_5 \\
z &= 3 - x_1 - x_4 - 2x_5
\end{align*}
\]

The coefficients of \(z\) are all negative. Hence the optimal solution is \(x_1 = x_4 = x_5 = 0, x_2 = x_3 = 1\) and it has cost 3.
2. [Duality]

(a) Give the Dual of the previous LP problem.
(b) Find the optimal dual solution, using whatever method you wish.
(c) Demonstrate the Complementary Slackness conditions for your optimal dual/primal solution pair. That is, what should be true and what is true for each of the 5 variable-inequality pairings.

The dual is:

\[
\begin{align*}
\text{minimize} & \quad 2y_1 + y_2 \\
\text{subject to:} & \\
& y_1 + y_2 \geq 1 \\
& y_1 + y_2 \geq 2 \\
& y_1 \geq 1
\end{align*}
\]

and \(y_1, y_2 \geq 0\). Two of the inequalities are redundant.

The solution \(y_1 = y_2 = 1\) is feasible and has a cost equal to the optimal primal solution. Hence this is the optimal dual solution.

Here are the five complementary slackness conditions:

(a) \(y_1 + y_2 > 1 \Rightarrow x_1 = 0\).
(b) \(x_2 \neq 0 \Rightarrow y_1 + y_2 = 2\).
(c) \(x_3 \neq 0 \Rightarrow y_1 = 1\).
(d) \(y_1 \neq 0 \Rightarrow x_1 + x_2 + x_3 = 2\).
(e) \(y_2 \neq 0 \Rightarrow x_1 + x_2 = 1\).

where the first three come from the primal variables, the last two come from the dual variables. The equalities are all satisfied.
3. [LU Decomposition]

(a) Use Gaussian Elimination with partial pivoting to decompose,

\[ A = \begin{bmatrix}
  1 & 1 & 1 \\
  2 & 0 & 2 \\
  0 & 3 & 3 \\
\end{bmatrix}, \]

into the product,

\[ L_3 P_3 L_2 P_2 L_1 P_1 A = U, \]

where \( L_i \) are column \( i \) eta-matrices, \( P_i \) are permutation matrices, and \( U \) is upper triangular with 1’s down the diagonal.

(b) Use back substitution and your decomposition to find \( x_1, x_2, x_3 \) real numbers which satisfy,

\[
\begin{align*}
  x_1 + x_2 + x_3 &= 1 \\
  x_1 + x_3 &= 1/2 \\
  x_2 + x_3 &= 1/3
\end{align*}
\]

\[
\begin{align*}
  &P_1 = \begin{bmatrix}
  0 & 1 & 0 \\
  1 & 0 & 0 \\
  0 & 0 & 1 \\
\end{bmatrix} & L_1 = \begin{bmatrix}
  1/2 & 0 & 0 \\
  -1/2 & 1 & 0 \\
  0 & 0 & 1 \\
\end{bmatrix} \\
  &P_2 = \begin{bmatrix}
  1 & 0 & 0 \\
  0 & 0 & 1 \\
  0 & 1 & 0 \\
\end{bmatrix} & L_2 = \begin{bmatrix}
  1 & 0 & 0 \\
  0 & 1/3 & 0 \\
  0 & -1/3 & 1 \\
\end{bmatrix} \\
  &P_3 = \begin{bmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1 \\
\end{bmatrix} & L_3 = \begin{bmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & -1 \\
\end{bmatrix} \\
  &U = \begin{bmatrix}
  1 & 0 & 1 \\
  0 & 1 & 1 \\
  0 & 0 & 1 \\
\end{bmatrix}
\]

Part (b) seeks \( x \) such that \( Ax = [1, 1, 1] \) (a column vector of 1’s). First calculate:

\[
L_3 P_3 L_2 P_2 L_1 P_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = L_3 P_3 L_2 P_2 \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix}
\]
\[
L_3P_3L_2 \begin{bmatrix} 1/2 \\ 1 \\ 1/2 \end{bmatrix} = L_3P_3 \begin{bmatrix} 1/2 \\ 1/3 \\ 1/6 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/3 \\ -1/6 \end{bmatrix}
\]

Then back substitute:

\[
Ux = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 1/2 \\ 1/3 \\ -1/6 \end{bmatrix}
\]
giving,

\[
x = \begin{bmatrix} 2/3 \\ 1/2 \\ -1/6 \end{bmatrix}.
\]
4. [Theory]

Prove that the product $AB$ of two square matrices is nonsingular if and only if both $A$ and $B$ are nonsingular.

Suppose $A$ and $B$ are nonsingular. By Theorem 6.2, a matrix is nonsingular if and only if there exists an inverse. In our case, there exist $A^{-1}$ and $B^{-1}$ such that,

$$AA^{-1} = A^{-1}A = B^{-1}B = B^{-1}B = I.$$ 

Then,

$$AB(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI A^{-1} = AA^{-1} = I,$$

and,

$$(B^{-1}A^{-1})AB = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I.$$ 

So $AB$ has an inverse, namely $B^{-1}A^{-1}$, and is therefore nonsingular.

Now suppose that $AB$ is nonsingular. By Theorem 6.1, a square matrix has two possibilities only: it is nonsingular and each equation $Ax = b$ has a unique solution for $x$, or it is singular and the equation $Ax = b$ has either no or an infinity of solutions, depending on the $b$. It is enough to show that $Ax = 0$ and $Bx = 0$ have unique solutions. Because $Ax = 0$ has a solution, $x = 0$, if it is singular it must have an infinity of solutions $Ax = 0$. If it has only one solution, than it must be nonsingular. The same goes for $B$.

Suppose $Bx_1 = Bx_2 = 0$. Then $ABx_1 = ABx_2 = 0$ so $x_1 = x_2 = 0$. Thus $B$ is nonsingular. If $Ax_1 = Ax_2 = 0$, then since $B$ is non-singular, there are unique solutions to $By_i = x_i$, with $i = 1, 2$. Hence $ABy_i = 0$ and thus $y_1 = y_2$, and $x_1 = By_1 = By_2 = x_2$. So $A$ is nonsingular.