RSA AND PROBABILISTIC FACTORING

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1. Introduction

There are notes for the Introduction to Cryptography course. This is the mathematical background to the RSA cryptosystem including an RP algorithm for primality testing, and the reduction of breaking RSA to integer factorization.

2. Extended Euclidean Algorithm

For positive integers \( a, b \), let \( (a, b) \) be the greatest common divisor (GCD) of \( a \) and \( b \). That is, \( (a, b) \) is a number such that it divides both \( a \) and \( b \), and for \( d \) dividing both \( a \) and \( b \), \( d \) also divides \( (a, b) \):

\[(a, b)|a, b \text{ and } \forall d \text{ s.t. } d|a, b \text{ then } d|(a, b).\]

**Theorem 2.1.** The GCD of two integers is an integer linear combination of those integers,

\[\exists s, t \in \mathbb{Z} \text{ s.t. } sa + tb = (a, b).\]

We can prove this along the way of showing how to calculate the GCD.

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**Definition 2.1** (Euclidean Algorithm). Given integers $a, b$ such that $a > b > 0$, define a sequence of remainders:

\[
\begin{align*}
r_0 &= a, \\
r_1 &= b, \\
r_2 &= r_0 \% r_1, \\
\cdots \\
r_i &= r_{i-2} \% r_{i-1}, \\
\cdots \\
d &= r_{K-1} \% r_K, \\
0 &= r_K \% d
\end{align*}
\]

where $x \% y$ is the remainder when $y$ divides $x$.

**Theorem 2.2.** Given the Euclidean Algorithm, as defined above, then $d$ is the GCD of $a$ and $b$.

My favorite proof of this uses the definition of an ideal.

**Definition 2.2** (Ideal). Given integers $a, b$, the ideal generated by $a$ and $b$ is the set of all integer linear combinations of $a$ and $b$:

\[
\langle a, b \rangle = \{ sa + tb | s, t \in \mathbb{Z} \}
\]

**Proof.** I leave it to the reader to prove that, for $a > b$,

\[
\langle a, b \rangle = \langle b, a \% b \rangle.
\]

It follows immediately that,

\[
\langle a, b \rangle = \langle d, 0 \rangle
\]

where $d$ is as given by the Euclidean Algorithm, and the ideal is in fact all multiples of $d$.

Since $a, b \in \langle a, b \rangle$, then $d|a, b$ therefor $d|(a, b)$. Since $(a, b)|a, b$, then $(a, b)$ divides all integer linear combinations of $a$ and $b$, hence $\forall x \in \langle a, b \rangle$ then $(a, b)|x$. Since $d \in \langle a, b \rangle$ then $(a, b)|d$. So $d = (a, b)$. \qed

The Extended Euclidean Algorithm back-substitutes in the Euclidean Algorithm to calculate the $s$ and $t$ promised by the first theorem.

**Algorithm 2.1** (Extended Euclidean Algorithm). Given the $r_i$ and $K$ as in the above Euclidean Algorithm, we have the basis:

\[
d = r_{K-1} - q_K r_K.
\]
Given the induction hypothesis:

\[ d = s_{i-1} r_{i-1} - t_{i-1} r_i \]

the induction step substitutes \( r_i = r_{i-2} - q_{i-1} r_{i-1} \),

\[
\begin{align*}
    d &= s_{i-1} r_{i-1} + t_{i-1} (r_{i-2} - q_{i-1} r_{i-1}) \\
    &= t_{i-1} r_{i-2} + (s_{i-1} - t_{i-1} q_{i-1}) r_{i-1} \\
    &= s_{i-2} r_{i-2} - t_{i-2} r_{i-1}.
\end{align*}
\]

and can therefore by induction derive,

\[ d = s_1 r_1 - t_1 r_0 = t b + s a. \]

3. Square roots of unity in \( Z_n \) and Miller-Rabin

Let \( n \) be an integer, and \( Z_n \) be the ring of integers modulo \( n \). We consider all \( x \) such that \( x^2 = 1 \) (mod \( n \)).

**Definition 3.1** (Nontrivial roots of unity). In \( Z_n^* \), the solutions to \( x^2 = 1 \) (mod \( n \)) are called the roots of unity. If the roots are \( \pm 1 \), then they are the trivial roots of unity. Else they are the non-trivial square roots of unity.

**Theorem 3.1.** Let \( n \) be an odd integer. If \( Z_n^* \) there are always the trivial roots of unity. If \( n \) is not a prime nor a prime power; else there are also non-trivial roots of unity.

**Proof.** Since \( x^2 = 1 \) in \( Z \) for \( x = \pm 1 \), it is true also for \( Z_n^* \).

If \( n \) is a prime or a prime power, let \( p | n \) be the prime. Then \( p \) cannot divide both \( (x - 1) \) or \( (x + 1) \). Since

\[ x^2 - 1 = (x - 1)(x + 1), \]

then a square root of unity mod \( n \) implies,

\[ p | (x - 1)(x + 1). \]

Hence either \( n | (x + 1) \) or \( n | (x - 1) \) and so \( x = \pm 1 \) (mod \( n \)).

Conversely, if \( n \) is not a prime or prime power, let \( n = a b \) with \( (a, b) = 1 \). Then there exists \( s \) and \( t \) such that \( = s a + t b = 1 \). Note,

\[ (s a - t b)^2 = (s a)^2 + (t b)^2 = (s a + t b)^2 = 1 \pmod{n} \]

and therefore \( s a - t b \neq 1 \pmod{n} \) is a nontrivial square root of unity. \( \square \)

Therefore we can demonstrate that \( n \) is composite without having to factor \( n \) through the number theoretic properties of \( n \) in at least two ways.

**Definition 3.2** (Witness to non-primality). Given \( n \). A number \( w \) relatively prime to \( n \) is a witness of non-primality of \( n \) if either,
In the first case, $w$ is called a weak witness. In the second case, $w$ is called a strong witness.

It turns out that if $n$ is not prime, a random $x$ as a fair chance of being a witness to primality.

**Algorithm 3.1** (Miller-Rabin). Choose a $w \in \mathbb{Z}_n^*$ at random. Do this by picking a non-zero $w \in \mathbb{Z}_n$ at random and checking that $(w, n) = 1$. Else $w$ is a witness by way of having factored $n$ “by mistake”.

Check if $w^{n-1} \not\equiv 1 \pmod{n}$. If so, $w$ is a weakness witness. Else write

$$n - 1 = s 2^u,$$

with $s$ odd and set $v = w^s \pmod{n}$ and iteratively replace $v$ with $v^2 \pmod{n}$ stopping when $v \equiv 1 \pmod{n}$. If the value before the final squaring is -1, the test is inconclusive. Else $w$ is a strong witness to non-primality.

This shows that problem of primality is co-RP. A problem is co-RP if non-membership (i.e. being a composite) has a witness that can be found in polynomial time with high probability. An algorithm for a problem in co-RP never concludes falsely against the proposition. In this case, if the algorithm concludes that $n$ is not a prime it has a witness as definitive proof. However, with vanishing probability the algorithm can be in error in acclaiming the statement (i.e. that $n$ is a prime), as it only has as proof not having found a witness otherwise.

Note that a strong witness allows $n$ to be factored.

**Theorem 3.2.** Let $\epsilon$ be a nontrivial square root of unity in $\mathbb{Z}_n^*$, with $n$ odd. Then $(\epsilon - 1, n)$ and $(\epsilon + 1, n)$ give nontrivial factors of $n$.

**Proof.** Since,

$$(\epsilon - 1)(\epsilon + 1) = \epsilon^2 - 1 = 0 \pmod{n}$$

and prime dividing both $\epsilon - 1$ and $\epsilon + 1$ is even. So each prime dividing $n$ is either due to $\epsilon - 1$ or $\epsilon + 1$. Therefore $(n, \epsilon + 1)$ will be a nontrivial factor of $n$ consisting of some of the primes at the full power in $n$, and $(n, \epsilon - 1)$ will be the remaining of the primes, again at their full power. \(\square\)

For an odd $n$, the number of square roots of unity will be $2^r$ where $r$ is the number of distinct primes that divide $n$. This is best seen by the isomorphism,

$$\mathbb{Z}_n \cong \prod_{p^\alpha \mid n} \mathbb{Z}_{p^\alpha}.$$
But 2, no pun intended, is odd. While $\mathbb{Z}_p$ has exactly two roots of unity, $\mathbb{Z}_2$ has only 1 root of unity, $\mathbb{Z}_4$ has 2, and all $\mathbb{Z}_{2^j}$ for $j > 2$ have 4 roots of unity.

Example: The first Carmichael number $561 = 3 \cdot 11 \cdot 17$ has the 8 roots of unity $1, 67, 188, 254, 307, 373, 494, 560$. As an example, $(561, 66) = 3 \cdot 11$ and $(561, 68) = 17$.

When $n$ is even, things are different.

4. RSA AND THE HARDNESS OF FACTORING

The RSA cryptosystem depends on the difficulty of given an $n$ and an $e$, and $n$ is of the form $n = pq$ where $p$ and $q$ are distinct primes, finding a $d$ such that $x^{ed} = x \pmod{n}$.

**Definition 4.1** (RSA public key cryptography). Let $p, q$ be distinct odd primes and $n = pq$. Let $e$ be an element of $\mathbb{Z}_{\phi(n)}^*$, where $\phi$ is the Euler phi function, and $d$ the inverse of $e \pmod{\phi(n)}$,

$$ed = 1 \pmod{\phi(n)}.$$ 

Then for any message $m \in \mathbb{Z}_n^*$, the encryption of $m$ by public key $e$ is,

$$E_e(m) = m^e \pmod{n}$$

and the decryption of message $c \in \mathbb{Z}_n^*$ by secret key $d$ is,

$$D_d(c) = c^d \pmod{n}.$$

**Lemma 4.1.**

$$D_d(E_d(m)) = m \pmod{n}$$

**Proof.**

$$D_d(E_d(m)) = m^{de} = m^{k\phi(n)+1} = m^{k\phi(n)}m = 1^km = m \pmod{n}.$$ 

Knowing the value of $\phi(n)$, the inverse of $e \pmod{\phi(n)}$ is efficiently computable using the Extended Euclidean Algorithm. However, knowledge of $\phi(n)$ is equivalent to factoring $n$. Consider the equation (over $\mathbb{Z}$),

$$x^2 - (n + 1 - \phi(n))x + n = x^2 - (p + q)x + pq = (x - p)(x - q) = 0.$$ 

It has roots $p$ and $q$, and the roots can be found in polynomial time using the quadratic formula.

Knowledge of $d$ is not exactly the same as knowledge of $\phi(n)$. However since,

$$ed - 1 = k\phi(n),$$

then,

$$x^{ed-1} = x^{k\phi(n)} = 1^k = 1 \pmod{n}.$$
for any \( x \in \mathbb{Z}_n^\ast \). This fact can be used with Miller-Rabin to force the appearance of a strong witness of non-primality for \( n \), which consequently can factor \( n \).

**Theorem 4.1.** Let \((e, p, q)\) be an RSA cryptosystem. With public key \((e, n = pq)\) and private key \(d = e^{-1} \pmod{(p - 1)(q - 1)}\). Let \( A(e, n) \rightarrow d \) be an Probabilistic Polynomial Time algorithm for extracting the secret key from the public key. Then there exists a Probabilistic Polynomial Time algorithm \( B^A(e, n) \rightarrow (p, q) \) that factors \( n \).

**Proof.** Run \( A(e, n) \) to get \( d \). Write \( ed - 1 = 2^us \) where \( s \) is odd. Choose a non-zero \( w \in \mathbb{Z}_n \) at random and check if \((n, x) = 1\), else stop with \( x \) a factor of \( n \). Set \( v = w^s \pmod{n} \) and repeatedly set \( v \) to \( v^2 \pmod{n} \) until it is 1. It must eventually be one because \( ed = k\phi(n) \). If a nontrivial root of unity appears, \( \epsilon^2 = 1 \pmod{n} \), with \( \epsilon \neq \pm1 \), factor \( n \) by \((n, \epsilon + 1)\) or \((n, \epsilon - 1)\).

Example: Let the primes be 7 and 11 so \( n = 77 \). Let \( e = 7 \), and \( \phi(77) = 6 \cdot 10 = 60 \). The Extended Euclidean algorithm gives,

\[
43 \cdot 7 + (-5) \cdot 60 = 1
\]

so \( 7^{-1} = 43 \pmod{60} \) and \( d = 43 \). Then,

\[
7 \cdot 43 - 1 = 300 = 2^2 \cdot 75.
\]

Let \( w \) be 29 (after a few choices). Then \( 29^{75} = 43 \), and \( 43^2 = 1 \). Then \((44, 77) = 11\) and \((42, 77) = 7\).