Proofs for Algorithms, 1

Burton J. Rosenberg
University of Miami

Theorem 1 For $x \in \mathbb{R}$ and $a, b \in \mathbb{Z}$, $a, b > 0$,

$$[x/ab] = \lfloor [x/a]/b \rfloor$$

and

$$[x/ab] = \lceil [x/a]/b \rceil.$$

Proof: To motivate the proof, consider the plane $\mathbb{R} \times \mathbb{R}$ with horizontal lines drawn at $y = 1$, $y = b$ and $y = ab$ and rays drawn from the origin passing through the integer points on $y = 1$, that is, $(i, 1)$, $i = 1, 2, \ldots$. These rays also pass through some integer points on $y = b$ and $y = ab$.

The set of all $x$ on the line $y = ab$ such that $\lceil x/ab \rceil = k$ lies inside the cone describe by rays from the origin passing through $((k - 1)ab, ab)$ and $(kab, ab)$, including the rightmost ray but excluding the leftmost ray. The intersection of this half open cone with the line $y = b$ gives all $x'$ such that $\lceil x'/b \rceil = k$.

The calculation $\lceil x/a \rceil$ follows the ray from the origin passing through $(x, ab)$ to its intersection with the $y = b$ line and then moving right along $y = b$ to the next integer point. Following the $(x, ab)$ ray leaves us within the half-open cone, and moving right to the next integer point does not leave the cone, since the ray through $(kab, ab)$ which is the righthand boundary of the cone also passes through the integer point $(kb, b)$.

We can follow this picture to state a formal proof:

$$
\begin{align*}
(k - 1)ab &< x \leq kab & \text{for some } k \in \mathbb{Z} \\
(k - 1)a &< x/b \leq ka & \text{use } b > 0 \\
(k - 1)a &< [x/b] \leq ka & \text{use } a \in \mathbb{Z} \\
(k - 1) &< [x/b]/a \leq k & \text{use } a > 0
\end{align*}
$$

hence $\lceil [x/b]/a \rceil = k$.

We did use that $a \in \mathbb{Z}$, and the following example shows that this is necessary. Let $a = b = \sqrt{2}$ and $x = 2$. Do the math. However, we did not need that $b \in \mathbb{Z}$, hence we have a slightly stronger result.

$\triangle$

Theorem 2 Let $a_1, \ldots, a_d \in \mathbb{R}$ and $a_d > 0$. Then,

$$\sum_{i=0}^{d} a_i x^i = \Theta(x^d).$$
**Proof:** For $x > 1$,
\[
\sum_{i=0}^{d-1} a_i x^i \leq \sum_{i=0}^{d-1} |a_i| x^i \leq x^{d-1} K
\]
where we have set
\[
K = \sum_{i=0}^{d-1} |a_i|
\]
for notational convenience. We likewise show,
\[
\sum_{i=0}^{d-1} a_i x^k \geq -x^{d-1} K.
\]
Hence,
\[
\sum_{i=0}^{d-1} a_i x^i = \Theta(x^d).
\]

**Theorem 3** For all $d, \epsilon > 0$, \(\log^n d n = o(n^\epsilon)\)

**Proof:** We first prove by induction the case $d \in \mathbb{Z}$. Basis $d = 1$. Apply L’Hospital’s rule to the indeterminate form,
\[
\lim_{n \to \infty} (\log n)/n^\epsilon = \lim_{n \to \infty} (1/n)/(en^{\epsilon-1}) = \epsilon/n^\epsilon \to 0.
\]
Hence for any $c > 0$ there is an $n_o$ such that for $n \geq n_o$, $(\log n)/n^\epsilon < c$. That is, $\log n = o(n^\epsilon)$.

Applying L’Hospital’s to the case of the general power,
\[
\lim_{n \to \infty} (\log^d n)/n^\epsilon = \lim_{n \to \infty} d(\log^{d-1} n)(1/n)/(en^{\epsilon-1}) = (d/e) \lim_{n \to \infty} (\log^{d-1} n)/n^\epsilon
\]
Hence $\log^d n = o(n^\epsilon)$ if $\log^{d-1} n = o(n^\epsilon)$, which complete the induction step. By the monotonicity in $d$ of $\log^d n$, the result extends to all $d \in \mathbb{R}, d > 0$. \(\triangle\)

**Theorem 4** For all $d > 0$ and $a > 1$, $n^d = o(a^n)$
Proof: The proof pattern is the same as the previous theorem. L'Hospital’s applied the case $d = 1$ gives,

$$\lim_{n \to \infty} \frac{n}{a^n} = \lim_{n \to \infty} \frac{1}{a^n \log a} = 0.$$ 

Again, use L’Hospital’s to prove the induction step and then extend to all real $d \geq 1$ by monotonicity. △

**Corollary 1** For all $\epsilon > 0$, $n \log n = o(n^{1+\epsilon})$.

Proof: This follows from a general result, if $f_1(n) = O(g_1(n))$ and $f_2(n) = o(g_2(n))$, then $f_1(n)f_2(n) = o(g_1(n)g_2(n))$. Applying this by setting $f_1(n) = n$ and $f_2(n) = \log n$. To show the general result, let $c_1$ and $n_1$ be such that, $f_1(n) < c_1 g_1(n)$ for all $n > n_1$

This is possible because $f_1(n) = O(g_1(n))$. Let $c > 0$ be chosen, and set $c_2 = c/c_1$. Since $f_2(n) = o(g_2(n))$, there exists an $n_2$ such that, $f_2(n) < c_2 g_2(n)$ for all $n > n_2$.

Multiplying the equalities (both are positive) and letting $n_o$ be the maximum of $n_1$ and $n_2$,

$$f_1(n)f_2(n) < (c_1 g_1(n))(c_2 g_2(n)) = c g_1(n)g_2(n) \text{ for all } n > n_o.$$ △